Diperna-Lions and Ambrosio's approach to ordinary differential equations

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Abstract

We shall give a concise and self-contained exposition on Diperna-Lions and Ambrosio's approach to ordinary differential equations with coefficient in Sobolev spaces. We focus on the case where the reference measure is the standard Gaussian measure on $\mathbb{R}^d$, the case having been studied by F. Cipriano and A.B. Cruzeiro.

1 Introduction

In 1983, Cruzeiro [6] proved on the Wiener space (W, H, μ) that for a vector field $A : W \to H$ in the Sobolev space $\mathbb{D}_\infty^2(W, H)$ if (i) $\int_W e^{\lambda(|A|_H + |\nabla A|_{H\otimes H} + |\mathrm{div}_\mu(A)|)} d\mu < +\infty$ for all $\lambda > 0$, then there exists a unique flow of measurable maps $U_t : W \to W$ such that $(U_t)_*\mu = K_t\, \mu$ and

$$U_t(x) = x + \int_0^t A(U_s(x))\, ds.$$

This result has been generalized later by several authors (see [4], [12]). But at the same time, she proved that on $\mathbb{R}^d$ if $A \in C^3$ and (ii) $\int_{\mathbb{R}^d} e^{\lambda_0(|A| + |\mathrm{div}_\gamma(A)|)} d\gamma < +\infty$ for some $\lambda_0 > 0$, then the similar results on $\mathbb{R}^d$ hold. In the latter case, the ordinary differential equation with coefficient A admits the unique solution up to the explosion time; so the condition (ii) insures the non-explosion and the existence of density. For the first case, even though A is smooth in the sense of Malliavin calculus, it is not in general continuous with respect to the Banach norm of W. In the procedure of smoothing, the estimate on $e^{|\nabla A|_{H\times H}}$ was used.

In 1989, R.J. Diperna and P.L. Lions [7] treated vector fields in $W^{1,1}_{loc}(\mathbb{R}^d)$. For the procedure of approximation, instead of estimating the Jacobian of vector fields, they transformed ordinary differential equations to transport equations. By establishing the well-posedness for these last ones, they solved ordinary differential equations. Later in 2004 [1], L. Ambrosio took advantage the use of continuity equations, which allowed him to construct a flow of quasi-invariant maps associated to vector fields Z with only BV regularity (see also [2]). In this spirit, Ambrosio and Figalli [3] proved recently the existence and uniqueness of the so-called L^r-regular flow associated to vector fields Z on abstract Wiener spaces, mainly under the low Sobolev regularity of Z and the integrability of $e^{\lambda \mathrm{div}_\mu(Z)}$ for large enough $\lambda > 0$. We refer to [14] for a general theory.

This note, based on [9], is to give a pedagogical and self-contained exposition to this theory.

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2 Flow of homeomorphisms under Osgood conditions

Let's consider first a vector field $V \in C_b^1(\mathbb{R}^d)$, having bounded derivative. Then the differential equation

$$\frac{dX_t}{dt} = V(X_t),\ X_0 = x \tag{2.1}$$

can be solved by Picard iteration and $x \to X_t(x)$ is a flow of global diffeomorphisms of $\mathbb{R}^d$; the inverse map $x \to X_t^{-1}(x)$ solves

$$\frac{dX_t^{-1}}{dt} = -V(X_t^{-1}),\ X_0^{-1} = x. \tag{2.2}$$

Now let $\theta \in C^1(\mathbb{R}^d)$. We denote by θ' the differential of θ, that is, for $x \in \mathbb{R}^d$, $\theta'(x)$ is a linear map from $\mathbb{R}^d$ to $\mathbb{R}$; we denote by $\nabla\theta$ the gradient of θ, that is a vector field on $\mathbb{R}^d$ such that $\nabla\theta(x)\cdot v = \theta'(x)\cdot v$ for $v \in \mathbb{R}^d$. For a vector field V on $\mathbb{R}^d$ and an application $F : \mathbb{R}^d \to \mathbb{R}^m$, we denote also by $D_V F$ the directional derivative of F along V, that is, $(D_V F)(x) = \frac{d}{dt}_{|t=0} F(X_t)$, where X_t is the flow associated to V. Let $u_t = \theta(X_t^{-1})$. Then

$$\frac{du_t}{dt} = \theta'(X_t^{-1})\cdot \frac{dX_t^{-1}}{dt}. \tag{i}$$

On the other hand,

$$\nabla u_t \cdot V = \theta'(X_t^{-1})\cdot D_V X_t^{-1}.$$

Now differentiating the equality $x = X_t^{-1}(X_t(x))$ with respect to the time t, we have

$$0 = \frac{dX_t^{-1}}{dt}(X_t) + (X_t^{-1})'(X_t)\frac{dX_t}{dt} = \frac{dX_t^{-1}}{dt}(X_t) + (D_V X_t^{-1})(X_t).$$

Since X_t is bijective, so for each $x \in \mathbb{R}^d$, the above equality gives

$$\frac{dX_t^{-1}}{dt} + D_V X_t^{-1} = 0.$$

According to (i), we get

$$\frac{du_t}{dt} + V\cdot\nabla u_t = 0, \quad u_0 = \theta \in C^1. \tag{2.3}$$

Conversely, if $u_t \in C^1$ is a solution to (2.3), then

$$\frac{d}{dt}[u_t(X_t)] = \frac{du_t}{dt}(X_t) + u_t'(X_t)\frac{dX_t}{dt} = \frac{du_t}{dt}(X_t) + (V\cdot\nabla u_t)(X_t) = 0,$$

so that $u_t(X_t) = \theta$ or $u_t = \theta(X_t^{-1})$: it is the unique solution to (2.3).

Now let $V : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded continuous vector field such that

$$|V(x) - V(y)| \le C|x-y|\log\frac{1}{|x-y|}, \quad \text{for } |x-y| \le \delta < 1. \tag{2.4}$$

Then the differential equation

$$\frac{dX_t}{dt} = V(X_t), \quad X_0 = x \tag{2.5}$$

admits a unique solution $(X_t)_{t\geq 0}$, which can be constructed by Euler approximation [10]. Choose $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi \geq 0, \quad \mathrm{supp}(\chi) \subset B(1), \quad \int_{\mathbb{R}^d} \chi dx = 1,$$

where $B(r) = \{x \in \mathbb{R}^d : |x| \leq r\}$. For $n \geq 1$, define $\chi_n(x) = 2^{dn}\chi(2^n x)$. Then $\mathrm{supp}(\chi_n) \subset B(2^{-n})$ and $\int_{\mathbb{R}^d} \chi_n dx = 1$. Set $V_n = V * \chi_n$ (convolution product), then V_n is a bounded smooth vector field on $\mathbb{R}^d$.

Proposition 2.1. *There exists $\zeta > 1$ such that*

$$\sup_{x\in\mathbb{R}^d} |V_n(x) - V(x)| \leq \zeta^{-n} \quad \text{for } n \text{ big enough.} \tag{2.6}$$

Proof. We have

$$\begin{aligned} |V_n(x) - V(x)| &\leq \int_{\mathbb{R}^d} |V(x-y) - V(y)|\chi_n(y)dy \leq C\int_{B(2^{-n})} |y| \log\frac{1}{|y|} \cdot \chi_n(y)dy \\ &\leq C2^{-n}\log 2^n \cdot \int_{B(2^{-n})} \chi_n(y)dy \leq C\zeta^{-n} \end{aligned}$$

for some $\zeta > 1$. □

Theorem 2.2. *Let $X_n(t,x)_{t\geq 0}$ be the solution to*

$$\frac{dX_n}{dt} = V_n(X_n), \quad X_n(0) = x.$$

Then for any $T > 0$,

$$\lim_{n\to\infty} \sup_{t\leq T} \sup_{x\in\mathbb{R}^d} |X_n(t,x) - X(t,x)| = 0. \tag{2.7}$$

Proof. For simplicity, we omit x in X_n as well as in X. Set $\xi_n(t) = |X_n(t) - X(t)|^2$ and

$$\tau_n = \inf\{t > 0 : \xi_n(t) \geq \delta^2\}.$$

Then by (2.4) and (2.6), for $t \leq \tau_n$,

$$\begin{aligned} \left|\frac{dX_n(t)}{dt} - \frac{dX(t)}{dt}\right| &\leq |V_n(X_n(t)) - V(X_n(t))| + |V(X_n(t)) - V(X(t))| \\ &\leq \zeta^{-n} + C|X_n(t) - X(t)|\log\frac{1}{|X_n(t) - X(t)|}. \end{aligned}$$

Therefore for $t \leq \tau_n$,

$$\left|\frac{d\xi_n(t)}{dt}\right| = 2\left|\langle X_n(t) - X(t), \frac{dX_n(t)}{dt} - \frac{dX(t)}{dt}\rangle\right| \leq 2\delta\zeta^{-n} + C\xi_n(t)\log\frac{1}{\xi_n(t)}.$$

Using the lemma below, we get for $t \leq \tau_n \wedge T$,

$$\xi_n(t) \leq (2\delta\zeta^{-n})^{e^{-Ct}} \leq (2\delta\zeta^{-n})^{e^{-CT}}.$$

This last quantity is less than δ^2 for n big enough. It follows that $\tau_n \geq T$ and

$$\sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}^d}\xi_n(t) \leq (2\delta\zeta^{-n})^{e^{-CT}}.$$

Letting $n \to \infty$ yields the result (2.7). □

A consequence of (2.7) is that $x \to X_t(x)$ defines a flow of homeomorphisms of $\mathbb{R}^d$. In fact, by previous section, the inverse maps X_n^{-1} as well as X_t^{-1} satisfy the same type differential equations. In the same way, X_n^{-1} converges to X_t^{-1} uniformly with respect to (t, x) in any compact subset of $[0, +\infty[\times\mathbb{R}^d$.

Lemma 2.3. *Let $\varphi : \mathbb{R}_+ \to (0, 1)$ be a derivable function such that for $C > 0$,*

$$\varphi'(t) \leq C\varphi(t)\log\frac{1}{\varphi(t)}, \tag{2.8}$$

then

$$\varphi(t) \leq (\varphi(0))^{e^{-Ct}} \quad \textit{for } t \geq 0.$$

Proof. $\log\varphi(t)$ being negative, we use (2.8),

$$\frac{\varphi'(t)}{\varphi(t)\log\varphi(t)} \geq -C.$$

Integrating this inequality between $(0, t)$, it leads to

$$\int_{\varphi(0)}^{\varphi(t)}\frac{ds}{s\log s} \geq -Ct, \quad \text{or} \quad \log\left(\frac{\log\varphi(t)}{\log\varphi(0)}\right) \geq -Ct.$$

Therefore $\log\varphi(t) \leq \log\varphi(0)\cdot e^{-Ct}$ or $\varphi(t) \leq (\varphi(0))^{e^{-Ct}}$. □

In order to apply Lemma 1.5 in the proof of Theorem 2.2, we set

$$\eta_n(t) = \frac{2\delta}{C}\zeta^{-n} + \xi_n(t)$$

and observe that for n big enough,

$$-C\xi_n(t)\log\xi_n(t) + 2\delta\zeta^{-n} \leq -C\eta_n(t)\log\eta_n(t).$$

□

Assume now the divergence $\text{div}(V) \in L^1_{loc}(\mathbb{R}^d)$ exists in the distribution sense:

$$\int_{\mathbb{R}^d}\text{div}(V)\,\varphi\,dx = -\int_{\mathbb{R}^d}\nabla\varphi\cdot V\,dx, \quad \varphi \in C_c^\infty(\mathbb{R}^d). \tag{2.9}$$

We have

$$\begin{aligned}\text{div}(V_n) &= \sum_{i=1}^d\frac{\partial}{\partial x_i}(V_n)^i = \sum_{i=1}^d\int_{\mathbb{R}^d}V^i(y)\frac{\partial}{\partial x_i}\chi_n(x-y)dy\\ &= -\int_{\mathbb{R}^d}V(y)\cdot\nabla_y(\chi_n(x-y))dy = \int_{\mathbb{R}^d}\text{div}(V)(y)\chi_n(x-y)dy = \text{div}(V)*\chi_n,\end{aligned}$$

It follows that $\text{div}(V_n)$ converges to $\text{div}(V)$ in $L^1_{loc}(\mathbb{R}^d)$, as $n \to +\infty$.

Theorem 2.4. *Assume* (2.4) *and* $\text{div}(V)$ *exists. Let* $\theta \in C(\mathbb{R}^d)$. *Then* $u_t(x) = \theta(X_t^{-1}(x))$ *satisfies the transport equation*

$$\frac{du_t}{dt} + V \cdot \nabla u_t = 0, \quad u_0 = \theta$$

in the sense that, for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$(u_t, \varphi)_{L^2} = (\theta, \varphi)_{L^2} + \int_0^t (u_s, \text{div}(\varphi V))_{L^2} ds. \tag{2.10}$$

Proof. Step 1. Suppose $\theta \in C^1$. Let X_n be given in Theorem 2.2 and set $u_n(t) = \theta(X_n^{-1}(t))$. Then for any $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$(u_n(t), \varphi)_{L^2} = (\theta, \varphi)_{L^2} + \int_0^t (u_n(s), \text{div}(\varphi V_n))_{L^2} ds. \tag{2.11}$$

Let K be the support of φ, then the support of $\text{div}(\varphi V_n) = \varphi\,\text{div}(V_n) + \nabla\varphi \cdot V_n$ is contained in K. Let $R = \sup_{0\leq t\leq T} \sup_{x\in K} |X^{-1}(t,x)|$ which is finite. By (2.7), for n big enough, $X_n^{-1}(t,x) \in B(R+1)$ for all $0 \leq t \leq T$, $x \in K$. Therefore for $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$,

$$\sup_{x\in K} \sup_{t\leq T} |\theta(X_n^{-1}(t,x)) - \theta(X^{-1}(t,x))| < \varepsilon.$$

Letting $n \to \infty$ in (2.11), we get (2.10).

Step 2. Let $\theta \in C(\mathbb{R}^d)$. Pick $\theta_n \in C^1(\mathbb{R}^d)$ such that θ_n converges to θ on any compact set. By Step 1, $u_t^n = \theta_n(X_t^{-1})$ satisfies (2.10). Now let $K = \text{supp}(\varphi)$. Then $\theta_n(X_t^{-1})$ converges uniformly to $\theta(X_t^{-1})$ over K. So that letting $n \to \infty$ in

$$(u_t^n, \varphi)_{L^2} = (\theta_n, \varphi)_{L^2} + \int_0^t (u_s^n, \text{div}(\varphi V))_{L^2} ds,$$

we get the result. □

Corollary 2.5. *If* $div(V) = 0$, *then* X_t *preserves the Lebesgue measure.*

Proof. Take $\theta \in C_c(\mathbb{R}^d)$. Let $K = \text{supp}(\theta)$. Then $K_T = \cup_{0\leq t\leq T} X_t(K)$, being the image of $[0,T] \times K$ under $(t,x) \to X_t(x)$, is compact. Now for $x \in (K_T)^c$, then for any $t \in [0,T]$, $X_t^{-1}(x) \in K^c$, so that $\theta(X_t^{-1}(x)) = 0$. Now take $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi = 1$ on K_T. We have for $s \leq t \leq T$,

$$(u_s, \text{div}(\varphi V))_{L^2} = \int u_s(x)\, \nabla\varphi \cdot V(x)\, dx = 0,$$

so that

$$\int_{\mathbb{R}^d} \theta(X_t^{-1}) dx = (u_t, \varphi)_{L^2} = (\theta, \varphi)_{L^2} = \int_{\mathbb{R}^d} \theta(x) dx,$$

which means that X_t^{-1} leaves the Lebesgue measure invariant, so does X_t □

For the general case, we have

Theorem 2.6. *Assume* $\text{div}(V) \in L^\infty$. *Then the Lebesgue measure* λ_d *on* $\mathbb{R}^d$ *is quasi-invariant under the flow* X_t: $(X_t)_*\lambda_d = k_t\, \lambda_d$; *moreover*

$$e^{-t||\text{div}(V)||_\infty} \leq k_t(x) \leq e^{t||\text{div}(V)||_\infty}. \tag{2.12}$$

Proof. Take a positive function $\theta \in C_c(\mathbb{R}^d)$ and set $u_t = \theta(X_t^{-1})$. As seen in the proof of the above corollary, there exists $R > 0$ such that $u_t(x) = 0$ for $t \in [0,T]$ and $|x| > R$. Then for $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi(x) = 1$ for $|x| \leq R$, we have, for $s \in [0,T]$, $(u_s, \mathrm{div}(\varphi V))_{L^2} = \int_{\mathbb{R}^d} u_s \,\mathrm{div}(V)\, dx$. The equation (2.10) yields

$$\int_{\mathbb{R}^d} u_t \, dx = \int_{\mathbb{R}^d} \theta \, dx + \int_0^t \Big(\int_{\mathbb{R}^d} u_s \,\mathrm{div}(V)\, dx \Big)\, ds.$$

The above equality says that $t \to \int_{\mathbb{R}^d} u_t \, dx$ is absolutely continuous and

$$\Big| \frac{d}{dt} \int_{\mathbb{R}^d} u_t \, dx \Big| \leq ||\mathrm{div}(V)||_\infty \int_{\mathbb{R}^d} u_t \, dx.$$

We deduce that

$$e^{-t||\mathrm{div}(V)||_\infty} \int_{\mathbb{R}^d} \theta \, dx \leq \int_{\mathbb{R}^d} \theta(X_t^{-1})\, dx \leq e^{t||\mathrm{div}(V)||_\infty} \int_{\mathbb{R}^d} \theta \, dx.$$

It follows that $(X_t^{-1})_* \lambda_d$ is absolutely continuous with respect to λ_d. Now (2.12) follows. □

3 Transport equations with vector fields in Sobolev spaces

Consider the standard Gaussian measure γ on $\mathbb{R}^d$. We denote by $\mathbb{D}_1^p(\mathbb{R}^d, \gamma)$ the Sobolev space, defined as the closure of C_b^∞ under the norm

$$||u||_{\mathbb{D}_1^p}^p = \int_{\mathbb{R}^d} |u(x)|^p \, d\gamma(x) + \int_{\mathbb{R}^d} |\nabla u(x)|^p \, d\gamma(x).$$

Contrary to the definition of divergence in the above section, we denote $\mathrm{div}_\gamma(Z)$ by

$$\int_{\mathbb{R}^d} \varphi \,\mathrm{div}_\gamma(Z)\, d\gamma = \int_{\mathbb{R}^d} \nabla\varphi \cdot Z \, d\gamma, \quad \varphi \in C_b^\infty.$$

It is known ([11]) that

$$||\mathrm{div}_\gamma(Z)||_{L^p(\gamma)} \leq C_p \, ||Z||_{\mathbb{D}_1^p}.$$

For $F \in C_b^1(\mathbb{R}^d)$, we denote

$$D_Z^* F = -Z \cdot \nabla F + F \mathrm{div}_\gamma(Z). \tag{3.1}$$

Then

$$\int_{\mathbb{R}^d} \nabla\psi \cdot Z F d\gamma = \int_{\mathbb{R}^d} \psi \, D_Z^* F \, d\gamma, \quad \psi \in C_b^\infty \tag{3.2}$$

In what follows, we fix $1 < p \leq 2$. Let $Z \in \mathbb{D}_1^p(\mathbb{R}^d, \mathbb{R}^d)$ be a vector field on $\mathbb{R}^d$. Let $T > 0$ be given.

Definition 3.1. *We say that* $u_\cdot \in L^\infty([0,T], L^q(\mathbb{R}^d, \gamma))$ *solves the transport equation*

$$\frac{du_t}{dt} + Z \cdot \nabla u_t = 0, \quad u|_{t=0} = u_0, \tag{3.3}$$

if for any $\alpha \in C_c^\infty([0,T))$ *and* $F \in C_b^\infty$*,*

$$\int_0^T \int_{\mathbb{R}^d} \big[-\alpha'(t) F u_t + \alpha(t)\, D_Z^* F\, u_t \big] d\gamma dt = \int_{\mathbb{R}^d} \alpha(0) F u_0 \, d\gamma, \tag{3.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Note that (3.4) holds for all $\alpha \in C_c^1([0,T))$. In what follows, we will discuss the existence and uniqueness of solutions to (3.3). First we prove the uniqueness under suitable conditions. Let $u_t \in L^q(\mathbb{R}^d)$ be a solution to (3.3), we consider $u_t^\varepsilon := P_\varepsilon u_t$, where P_ε is the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^d$, defined as follows:

$$P_\varepsilon F(x) = \int_{\mathbb{R}^d} F(e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon}} y) d\gamma(y), \tag{3.5}$$

which is in C_b^∞ if F is bounded. Similarly P_ε on vector valued functions is defined. We have

$$\nabla P_\varepsilon F = e^{-\varepsilon} P_\varepsilon(\nabla F), \quad \text{for } F \in C_b^1. \tag{3.6}$$

For any $F \in C_b^\infty(\mathbb{R}^d)$ and $\alpha \in C_c^\infty([0,T))$, we have,

$$\int_0^T \int_{\mathbb{R}^d} \alpha'(t) F u_t^\varepsilon \, d\gamma dt = \int_0^T \int_{\mathbb{R}^d} \alpha'(t) u_t P_\varepsilon F \, d\gamma dt.$$

By (3.4), this last term is equal to

$$\int_0^T \int_{\mathbb{R}^d} \alpha(t) D_Z^*(P_\varepsilon F)\, u_t \, d\gamma dt - \int_{\mathbb{R}^d} \alpha(0) u_0 P_\varepsilon F \, d\gamma. \tag{3.7}$$

But

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \alpha(t) D_Z^*(P_\varepsilon F)\, u_t \, d\gamma dt \\ =& \int_0^T \int_{\mathbb{R}^d} \alpha(t) u_t \left(P_\varepsilon(D_Z^* F) + D_Z^*(P_\varepsilon F) - P_\varepsilon(D_Z^* F) \right) d\gamma dt \\ =& \int_0^T \int_{\mathbb{R}^d} \alpha(t) u_t^\varepsilon D_Z^* F \, d\gamma dt - \int_0^T \int_{\mathbb{R}^d} \alpha(t) F B_\varepsilon(u_t, Z) \, d\gamma dt, \end{aligned} \tag{3.8}$$

where

$$B_\varepsilon(f, Z) = D_Z P_\varepsilon f - P_\varepsilon(D_Z f). \tag{3.9}$$

Therefore u_t^ε satisfies the transport equation

$$\frac{du_t^\varepsilon}{dt} + Z \cdot \nabla u_t^\varepsilon = B_\varepsilon(u_t, Z) \tag{3.10}$$

with the initial condition $P_\varepsilon u_0$. The following result will play a key role, whose proof is postponed in the last section.

Theorem 3.2. *For any* $1 < p \leq 2$*, it holds that*

$$||B_\varepsilon(f, Z)||_{L^1} \leq C_p \, ||f||_{L^q} ||Z||_{\mathbb{D}_1^p}. \tag{3.11}$$

Using the estimate (3.11), we have

Proposition 3.3.

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^d}|B_\varepsilon(u_t,Z)|d\gamma dt=0. \tag{3.12}$$

Proof. Note first that $||B_\varepsilon(f,Z)||_{L^1}\to 0$ as $\varepsilon\to 0$ for $f\in C^1$. Now the estimate (3.11) proves that $\lim_{\varepsilon\to 0}||B_\varepsilon(f,Z)||_{L^1}=0$ holds for $f\in L^q$. Then for $t\in[0,T]$, $\lim_{\varepsilon\to 0}\|B_\varepsilon(u_t,Z)\|_{L^1}=0$. But

$$\|B_\varepsilon(u_t,Z)\|_{L^1}\le C_p\,\|u_t\|_{L^q}\|Z\|_{\mathbb{D}_1^p}\le C_p\left(\sup_{t\in[0,T]}\|u_t\|_{L^q}\right)\|Z\|_{\mathbb{D}_1^p}.$$

Hence Lebesgue's dominated convergence theorem leads to

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^d}|B_\varepsilon(u_t,Z)|\,d\gamma dt=0.$$

□

Theorem 3.4. *(Uniqueness) Let $Z\in\mathbb{D}_1^p(\mathbb{R}^d,\gamma)$ with $1<p\le 2$. Assume that*

$$\int_{\mathbb{R}^d}\exp\left(\lambda_0|\mathrm{div}_\gamma(Z)|\right)d\gamma<+\infty,\quad \textit{for some } \lambda_0>0, \tag{3.13}$$

then the transport equation (3.3) *admits at most one solution $u\in L^\infty([0,T],L^q(\mathbb{R}^d))$ for the given $u_0\in L^q$, where $\frac{1}{p}+\frac{1}{q}=1$.*

Proof. By linearity, we assume that $u_0=0$. Let $u_t^\varepsilon=P_\varepsilon u_t$. It is known that

$$||\nabla P_\varepsilon f||_{L^p}\le C_{\varepsilon,p}\,||f||_{L^p}. \tag{3.14}$$

We have

$$\|Z\cdot\nabla u_t^\varepsilon\|_{L^1}\le\|Z\|_{L^p}\|\nabla u_t^\varepsilon\|_{L^q}\le C_{q,\varepsilon}\|Z\|_{L^p}\|u_t\|_{L^q}.$$

Therefore by (3.10), for γ-a.s. $x\in\mathbb{R}^d$, $t\to u_t^\varepsilon$ is absolutely continuous. Then for $\beta\in C_b^1(\mathbb{R})$, $\frac{d}{dt}\beta(u_t^\varepsilon)=\beta'(u_t^\varepsilon)\,\frac{d}{dt}u_t^\varepsilon$ and $Z\cdot\nabla(\beta(u_t^\varepsilon))=\beta'(u_t^\varepsilon)\,Z\cdot\nabla u_t^\varepsilon$. Thus

$$\begin{aligned}\frac{d}{dt}\beta(u_t^\varepsilon)+Z\cdot\nabla(\beta(u_t^\varepsilon))&=\beta'(u_t^\varepsilon)\left(\frac{d}{dt}u_t^\varepsilon+Z\cdot\nabla u_t^\varepsilon\right)\\&=\beta'(u_t^\varepsilon)\,B_\varepsilon(u_t,Z),\end{aligned}$$

or

$$\frac{d}{dt}\beta(u_t^\varepsilon)+Z\cdot\nabla(\beta(u_t^\varepsilon))=\beta'(u_t^\varepsilon)\,B_\varepsilon(u_t,Z). \tag{3.15}$$

Letting $\varepsilon\to 0$ in (3.15) gives

$$\frac{d}{dt}\beta(u_t)+Z\cdot\nabla\beta(u_t)=0. \tag{3.16}$$

For $F\in C_b^\infty(\mathbb{R}^d)$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}F\beta(u_t)\,d\gamma=\int_{\mathbb{R}^d}D_ZF\,\beta(u_t)\,d\gamma-\int_{\mathbb{R}^d}\mathrm{div}_\gamma(Z)F\beta(u_t)\,d\gamma \tag{3.17}$$

Taking $F = 1$ in (3.17), we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}\beta(u_t)d\gamma = -\int_{\mathbb{R}^d}\operatorname{div}_\gamma(Z)\beta(u_t)\,d\gamma. \tag{3.18}$$

After a smoothing procedure as in [7], we can take $\beta(s) = (|s|\wedge M)^q$ for $M > 0$ fixed. Following [5], for $R > 0$, we define

$$\Sigma_R = \{x \in \mathbb{R}^d;\ |\operatorname{div}_\gamma(Z)| \geq R\}.$$

Then for any $0 < \lambda \leq \lambda_0$, $\gamma(\Sigma_R) \leq e^{-\lambda R}\int_{\mathbb{R}^d} e^{\lambda\,|\operatorname{div}_\gamma(Z)|}d\gamma$ which is finite by condition (3.13). Denote by $C^2_{\lambda_0,q}$ this last integral for $\lambda = \lambda_0$. We have by Cauchy-Schwartz inequality,

$$\int_{\Sigma_R}|\operatorname{div}_\gamma(Z)|\,\beta(u_t)\,d\gamma \leq \|\beta\|_\infty\,\|\operatorname{div}_\gamma(Z)\|_{L^2}\,e^{-\lambda_0 R/2}C_{\lambda_0,q}.$$

Combining with (3.18), we get for some constant $C_{\lambda_0,M}$ independent of R,

$$\frac{d}{dt}\int_{\mathbb{R}^d}\beta(u_t)\,d\gamma \leq R\int_{\mathbb{R}^d}\beta(u_t)\,d\gamma + C_{\lambda_0,M}\,e^{-\lambda_0 R/2}.$$

By Gronwall lemma, we have for each $t \in [0, T]$,

$$\int_{\mathbb{R}^d}\beta(u_t)\,d\gamma \leq C_{\lambda,M}\,e^{-\lambda_0 R/2}e^{Rt},$$

which tends to 0 as $R \to +\infty$, for

$$t \leq T_0 < \lambda_0/2.$$

It follows that for $t \in [0, T_0]$,

$$|u_t| \wedge M = 0 \quad \text{a.s.}$$

Letting $M \to +\infty$, we see that $u_t = 0$ for $t \in [0, T_0]$. Now shifting the time, $u_t = 0$ for all $t \in [0, T]$. □

In order to construct solutions to the transport equation (3.3), we prepare

Proposition 3.5. *Let $B \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $(X_t)_{t\in\mathbb{R}}$ be the flow of diffeomorphisms associated to B:*

$$\frac{dX_t}{dt} = B(X_t), \quad X_0 = x.$$

Then $(X_t)_*\gamma = K_t\,\gamma$ with K_t satisfying

$$K_t = \exp\Bigl(\int_0^t \operatorname{div}_\gamma(B)(X_{-s})ds\Bigr), \quad ||K_t||^p_{L^p} \leq \int_{\mathbb{R}^d}\exp\Bigl(\frac{p^2 t}{p-1}|\operatorname{div}_\gamma(B)|\Bigr)\,d\gamma. \tag{3.19}$$

Proof. Note that $\operatorname{div}_\gamma(B) = \sum_{i=1}^d\bigl(x_iB_i(x) - \frac{\partial B_i}{\partial x_i}\bigr)$; therefore $e^{\lambda|\operatorname{div}_\gamma(B)|}$ is in $L^1(\gamma)$ for all $\lambda > 1$. Let $\varphi \in C_b^1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d}\varphi\,K_t\,d\gamma = \int_{\mathbb{R}^d}\varphi(X_t)\,d\gamma.$$

Taking the derivative with respect to t, at $t = 0$, we get

$$\Bigl(\frac{d}{dt}\int_{\mathbb{R}^d}\varphi\,K_t\,d\gamma\Bigr)_{|t=0} = \int_{\mathbb{R}^d}\nabla\varphi\cdot B\,d\gamma = \int_{\mathbb{R}^d}\varphi\operatorname{div}_\gamma(B)\,d\gamma.$$

It follows that

$$\Big(\frac{dK_t}{dt}\Big)_{|t=0} = \mathrm{div}_\gamma(B).$$

Now using the property of flow: $X_{t+s} = X_t \circ X_s$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi K_t\, d\gamma = \int_{\mathbb{R}^d} \varphi(X_t)\,\mathrm{div}_\gamma(B)\, d\gamma = \int_{\mathbb{R}^d} \varphi\,\mathrm{div}_\gamma(B)(X_{-t})K_t\, d\gamma.$$

Therefore

$$\frac{dK_t}{dt} = \mathrm{div}_\gamma(B)(X_{-t})\, K_t \quad K_0 = 1,$$

which gives the expression $K_t = e^{\int_0^t \mathrm{div}_\gamma(B)(X_{-s})ds}$. For estimating L^p norm of K_t, we set

$$I(t) = \sup_{|s|\leq t}\int_{\mathbb{R}^d} K_s^p(x)d\gamma(x), \quad t>0.$$

By Jensen inequality,

$$e^{p\int_0^t |\mathrm{div}_\gamma(B)(X_{-s})|ds} \leq \int_0^t e^{pt|\mathrm{div}_\gamma(B)(X_{-s})|}\,\frac{ds}{t}.$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^d} K_t^p\, d\gamma &\leq \int_0^t \Big(\int_{\mathbb{R}^d} e^{pt|\mathrm{div}_\gamma(B)|}\, K_{-s}\, d\gamma\Big)\frac{ds}{t}\\
&\leq \int_0^t \Big(\int_{\mathbb{R}^d} K_{-s}^p\, d\gamma\Big)^{1/p}\Big(\int_{\mathbb{R}^d} e^{\frac{p^2t}{p-1}|\mathrm{div}_\gamma(B)|}d\gamma\Big)^{(p-1)/p}\frac{ds}{t}\\
&\leq I(t)^{1/p}\Big(\int_{\mathbb{R}^d} e^{\frac{p^2t}{p-1}|\mathrm{div}_\gamma(B)|}d\gamma\Big)^{(p-1)/p}.
\end{aligned}$$

By changing B into $-B$, we get

$$\int_{\mathbb{R}^d} K_{-t}^p\, d\gamma \leq I(t)^{1/p}\Big(\int_{\mathbb{R}^d} e^{\frac{p^2t}{p-1}|\mathrm{div}_\gamma(B)|}d\gamma\Big)^{(p-1)/p}.$$

Combining these two inequalities, we get the second term in (3.19). $\square$

Now we are turning our attention to prove the existence of solutions to (3.3) in $L^\infty([0,T], L^q(\mathbb{R}^d))$. Let $\hat{u}_0 \in C_b^1(\mathbb{R}^d)$, and set $\hat{u}_t = \hat{u}_0(X_t^{-1})$. We saw that $\hat{u}_t$ solves the transport equation

$$\frac{d\hat{u}_t}{dt} + B\cdot\nabla\hat{u}_t = 0. \tag{3.20}$$

Let $q>1$ and $\bar{q}>q$,

$$||\hat{u}_t||_{L^q}^q = \int_{\mathbb{R}^d} |\hat{u}_0(X_t^{-1})|^q\, d\gamma = \int_{\mathbb{R}^d} |\hat{u}_0|^q K_{-t}\, d\gamma,$$

which is dominated by, using Hölder inequality and the second term in (3.19),

$$||\hat{u}_0||_{L^{\bar{q}}}^q\, ||K_{-t}||_{L^{q'}} \leq ||\hat{u}_0||_{L^{\bar{q}}}^q \Big[\int_{\mathbb{R}^d} \exp\Big(\frac{\bar{q}^2 t}{q(\bar{q}-q)}|\mathrm{div}_\gamma(B)|\Big)\, d\gamma\Big]^{(\bar{q}-q)/\bar{q}},$$

where $q' = \frac{\bar q}{\bar q - q}$. So we get

$$||\hat u_t||_{L^q} \le ||\hat u_0||_{L^{\bar q}} \Big[\int_{\mathbb{R}^d} \exp\Big(\frac{\bar q^2 t}{q(\bar q - q)}|\mathrm{div}_\gamma(B)|\Big)\, d\gamma\Big]^{(\bar q - q)/q\bar q}. \tag{3.21}$$

Next consider a vector field $Z \in \mathbb{D}_1^p(\mathbb{R}^d, \gamma)$. Let P_ε be the Ornstein-Uhlenbeck semi-group acting on vector fields. For $\varphi \in C_b^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \nabla\varphi \cdot P_\varepsilon Z\, d\gamma = \int_{\mathbb{R}^d} P_\varepsilon(\nabla\varphi)\cdot Z\, d\gamma,$$

which is equal to, according to (3.6),

$$e^\varepsilon \int_{\mathbb{R}^d} \nabla P_\varepsilon \varphi \cdot Z\, d\gamma = e^\varepsilon \int_{\mathbb{R}^d} \varphi\, P_\varepsilon(\mathrm{div}_\gamma(Z))\, d\gamma.$$

It follows that

$$\mathrm{div}_\gamma(P_\varepsilon Z) = e^\varepsilon P_\varepsilon \mathrm{div}_\gamma(Z). \tag{3.22}$$

Note that $P_\varepsilon Z \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, but not necessarily bounded with its derivatives. For $N \ge 1$, consider the cut-off function $\varphi_N \in C_c^\infty(\mathbb{R}^d)$ such that $0 \le \varphi_N \le 1$ and for $|x| \le N$, $\varphi_N(x) = 1$; for $|x| > N+2$, $\varphi_N(x) = 0$ and $\|\nabla\varphi_N\|_\infty \le 1$. Then $(P_\varepsilon Z)\varphi_N \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$. We have

$$\begin{aligned}\mathrm{div}_\gamma\big((P_\varepsilon Z)\varphi_N\big) &= \varphi_N \mathrm{div}_\gamma(P_\varepsilon Z) - \nabla\varphi_N \cdot P_\varepsilon Z\\ &= e^\varepsilon \varphi_N\, P_\varepsilon \mathrm{div}_\gamma(Z) - \nabla\varphi_N \cdot P_\varepsilon Z.\end{aligned} \tag{3.23}$$

It follows that

$$\big|\mathrm{div}_\gamma((P_\varepsilon Z)\,\varphi_N)\big| \le e^\varepsilon P_\varepsilon|\mathrm{div}_\gamma(Z)| + P_\varepsilon|Z|. \tag{3.24}$$

By Jensen inequality and the invariance of γ under P_ε, we have

$$\int_{\mathbb{R}^d} \exp\big(2e^\varepsilon P_\varepsilon|\mathrm{div}_\gamma(Z)|\big)\, d\gamma \le \int_{\mathbb{R}^d} \exp\big(2e^\varepsilon|\mathrm{div}_\gamma(Z)|\big)\, d\gamma.$$

Now using (3.24) and Cauchy-Schwarz inequality, we have the bound

$$\int_{\mathbb{R}^d} e^{\lambda|\mathrm{div}_\gamma((P_\varepsilon Z)\varphi_N)|} d\gamma \le \Big(\int_{\mathbb{R}^d} \exp\big(2e^\varepsilon\lambda|\mathrm{div}_\gamma(Z)|\big)\, d\gamma\Big)^{1/2}\Big(\int_{\mathbb{R}^d} e^{2\lambda|Z|} d\gamma\Big)^{1/2}. \tag{3.25}$$

Now we assume that

$$\int_{\mathbb{R}^d} e^{\lambda(|Z| + |\mathrm{div}_\gamma(Z)|)} < +\infty \quad \text{for all } \lambda > 0. \tag{3.26}$$

We will take a sequence $\varepsilon_m \to 0$ and $N_m \to +\infty$, and define $B_m = (P_{\varepsilon_m} Z)\varphi_{N_m}$. By (3.23), as $m \to +\infty$, in L^p,

$$B_m \to Z \quad \text{and} \quad \mathrm{div}_\gamma(B_m) \to \mathrm{div}_\gamma(Z). \tag{3.27}$$

Applying (3.21) to B_m and we denote by $\hat u_t^m$ the associated solution. We have

$$\|\hat u_t^m\|_{L^q} \le C_T\, \|\hat u_0^m\|_{L^{\bar q}} \left[\int_{\mathbb{R}^d} \exp\left(\frac{\bar q^2 T}{q(\bar q - q)}|\mathrm{div}_\gamma(B_m)|\right) d\gamma\right]^{(\bar q - q)/q\bar q}. \tag{3.28}$$

Combining with (3.25), we see that $\{\hat{u}^m;\ m \geq 1\}$ is bounded in $L^q([0,T]\times\mathbb{R}^d)$; therefore up to a subsequence, $\hat{u}^m$ converges to $u \in L^q([0,T]\times\mathbb{R}^d)$ weakly as $m \to +\infty$:

$$\int_0^T\int_{\mathbb{R}^d} u_t\psi(t,x)\,d\gamma(x)dt = \lim_{m\to+\infty}\int_0^T\int_{\mathbb{R}^d}\hat{u}_t^m(x)\psi(t,x)\,d\gamma dt, \quad \psi\in L^{q'}([0,T]\times\mathbb{R}^d),$$

where q' is the conjugate number of q : $\frac{1}{q}+\frac{1}{q'}=1$. Since $\mathrm{div}_\gamma(B_m)$ converges to $\mathrm{div}_\gamma(Z)$ in L^p, again up to a subsequence, $\mathrm{div}_\gamma(B_m)$ converges to $\mathrm{div}_\gamma(Z)$ a.s. Using the bound in (3.25), as $m \to +\infty$,

$$\int_{\mathbb{R}^d}\exp\left(\frac{\bar{q}^2T}{q(\bar{q}-q)}|\mathrm{div}_\gamma(B_m)|\right)d\gamma \to \int_{\mathbb{R}^d}\exp\left(\frac{\bar{q}^2T}{q(\bar{q}-q)}|\mathrm{div}_\gamma(Z)|\right)d\gamma.$$

Denote this last term by $A^{q\bar{q}/(\bar{q}-q)}$. By (3.28),

$$\limsup_{m\to+\infty}\|\hat{u}_t^m\|_{L^q} \leq C_T\,\|u_0\|_{L^{\bar{q}}}\,A.$$

Now consider $\psi(t,x)=\alpha(t)\varphi(x)$ with $\alpha\in L^{q'}([0,T])$ and $\varphi\in L^{q'}(\mathbb{R}^d)$, then

$$\left|\int_0^T\int_{\mathbb{R}^d}\alpha(t)\varphi(x)\,u_t(x)d\gamma(x)dt\right| \leq \|\alpha\|_{L^1}\|\varphi\|_{L^{q'}}\,C_T\,\|u_0\|_{L^{\bar{q}}}\,A.$$

It follows that for a.s. $t\in[0,T]$, $u_t\in L^q(\mathbb{R}^d)$ and $\|u_t\|_{L^q}\leq C_T\,\|u_0\|_{L^{\bar{q}}}\,A$; or explicitly

$$\|u_t\|_{L^q} \leq C_T\,\|u_0\|_{L^{\bar{q}}}\left[\int_{\mathbb{R}^d}\exp\left(\frac{\bar{q}^2T}{q(\bar{q}-q)}|\mathrm{div}_\gamma(Z)|\right)d\gamma\right]^{(\bar{q}-q)/q\bar{q}}. \tag{3.29}$$

Now it is easy to check that u_t solves the transport equation

$$\frac{du_t}{dt}+Z\cdot\nabla u_t=0, \quad u_0\in L^{\bar{q}}\text{ given.}$$

We state now the result that we have obtained:

Theorem 3.6. *Let $Z\in\mathbb{D}_1^p(\mathbb{R}^d,\gamma)$ be a vector field and*

$$\int_{\mathbb{R}^d} e^{\lambda(|Z|+|\mathrm{div}_\gamma(Z)|)}d\gamma < +\infty, \quad \textit{for each } \lambda>1. \tag{3.30}$$

Then for any $u_0\in L^{q+\varepsilon}(\gamma)$, the transport equation (3.3) has a solution $u\in L^\infty([0,T],L^q(\mathbb{R}^d))$.

4 Flows of quasi-invariant maps

In this section, we will consider the following condition

$$\int_{\mathbb{R}^d} e^{\lambda_0(|Z|+|\mathrm{div}_\gamma(Z)|)}d\gamma < +\infty \quad \text{for some small } \lambda_0>0. \tag{4.1}$$

Let $Z \in \mathbb{D}_1^p(\mathbb{R}^d, \gamma)$ and consider $B_m = P_{\varepsilon_m} Z \,\varphi_{N_m}$ defined in section 3. Recall that

$$\int_{\mathbb{R}^d} e^{\lambda|\mathrm{div}_\gamma(B_m)|} d\gamma \le \Big(\int_{\mathbb{R}^d} \exp(2e\lambda|\mathrm{div}_\gamma(Z)|)\, d\gamma\Big)^{1/2} \Big(\int_{\mathbb{R}^d} e^{2\lambda|Z|} d\gamma\Big)^{1/2}, \tag{4.2}$$

which is finite for

$$\lambda \le \frac{\lambda_0}{2e}.$$

Let $(X_t^m)_{t\in\mathbb{R}}$ solve

$$\frac{dX_t^m}{dt} = B_m(X_t^m), \quad X_0^m = x,$$

and K_t^m be the density of $(X_t^m)_\gamma$ with respect to γ. By second term in (3.19), for any $q > 1$, there is a small $T_0 = \frac{q-1}{q}\frac{\lambda_0}{2e}$ such that for $t \in [0, T_0]$,

$$||K_t^m||_{L^q}^q \le \int_{\mathbb{R}^d} \exp\Big(\frac{q^2 t}{q-1}|\mathrm{div}_\gamma(B_m)|\Big) d\gamma, \tag{4.3}$$

which is dominated by a constant independent of m, according to (4.2).

Proposition 4.1. *Let $T > 0$ be given. Then for $t \in [0, T]$,*

$$\int_{\mathbb{R}^d} K_t^m |\log K_t^m|\, d\gamma \le T\, ||\mathrm{div}_\gamma(B_m)||_{L^{2^N}} \int_{\mathbb{R}^d} e^{\frac{\lambda_0}{2e}|\mathrm{div}_\gamma(B_m)|}\, d\gamma, \tag{4.4}$$

where $NT_0 \ge T$ and $T_0 \le \frac{\lambda_0}{8e}$.

Proof. Using the explicit expression given in (3.19), we have

$$\log K_t^m(X_t^m) = \int_0^t \mathrm{div}_\gamma(B_m)(X_{t-s}^m)\, ds.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^d} K_t^m |\log K_t^m|\, d\gamma &= \int_{\mathbb{R}^d} |\log K_t^m(X_t^m)|\, d\gamma \\ &\le \int_0^t \int_{\mathbb{R}^d} |\mathrm{div}_\gamma(B_m)(X_{t-s}^m)|\, d\gamma\, ds.\end{aligned}$$

Note by (4.3) for $q = 2$,

$$||K_{T_0}^m||_{L^2}^2 \le \int_{\mathbb{R}^d} e^{4T_0|\mathrm{div}_\gamma(B_m)|}\, d\gamma.$$

Using the property of flow, for $0 \le t \le T_0$:

$$\int_{\mathbb{R}^d} |\mathrm{div}_\gamma(B_m)(X_{t+T_0}^m)|\, d\gamma = \int_{\mathbb{R}^d} |\mathrm{div}_\gamma(B_m)(X_t^m)\, K_{T_0}^m\, d\gamma,$$

which is less, by Cauchy-Schwartz inequality than

$$\Big(\int_{\mathbb{R}^d} |\mathrm{div}_\gamma(B_m)(X_t^m)|^2\, d\gamma\Big)^{1/2} ||K_{T_0}^m||_{L^2},$$

again by Cauchy-Schwartz inequality, less than

$$||\mathrm{div}_\gamma(B_m)||_{L^4}||K_t^m||_{L^2}^{1/2}\,||K_{T_0}^m||_{L^2}.$$

Set

$$A(m)=\sup_{0\le t\le T}||K_t^m||_{L^2}^2,\quad A=\sup_{m\ge 1}A(m). \tag{4.5}$$

Therefore by induction, we get for $t\in[0,T]$,

$$\int_{\mathbb{R}^d}|\mathrm{div}_\gamma(B_m)(X_t^m)|\,d\gamma\le ||\mathrm{div}_\gamma(B_m)||_{L^{2^N}}\,A(m)^{1/2^N}\cdots A(m)^{1/2}\le ||\mathrm{div}_\gamma(B_m)||_{L^{2^N}}\,A(m),$$

where N is such that $NY_0\ge T$. Now the estimate (4.3) leads to (4.4). □

Let $\{\ell_1,\cdots,\ell_d\}$ be the dual basis of $\mathbb{R}^d$. Consider $u_i^m(t,x)=\ell_i(X_t^m(x))$. Then we saw that $\{u_i^m;\ t\in[0,T]\}$ solves the transport equations

$$\frac{du_i^m}{dt}-B_m\cdot\nabla u_i^m=0,\quad u_i^m(0)=\ell_i. \tag{4.6}$$

Proposition 4.2. *Let $T>0$ be given and any $q>1$. Then there exists a constant $C_{\lambda_0,T,q}>0$ independent of m such that*

$$\int_{\mathbb{R}^d}|u_i^m(t,x)|^q\,d\gamma(x)\le C_{\lambda_0,T,q},\quad t\in[0,T]. \tag{4.7}$$

Proof. First for $t\in[0,T_0]$, we have $\int_{\mathbb{R}^d}|u_i^m(t)|^q\,d\gamma=\int_{\mathbb{R}^d}|\ell_i(x)|^qK_t^m\,d\gamma$, which is dominated by

$$|||\ell_i|^q||_{L^2}||K_t^m||_{L^2}\le A(m)^{1/2}\,|||\ell_i|^q||_{L^2}.$$

Again by property of flow and by induction, we have

$$\int_{\mathbb{R}^d}|\ell_i(X_t^m)|^q\,d\gamma\le\Bigl(\int_{\mathbb{R}^d}|\ell_i(x)|^{q2^N}\,d\gamma\Bigr)^{2^{-N}}\cdot A(m)^{2^{-N}}\cdots A(m)^{1/2}\le A(m)\,||\ell_i||_{L^{q2^N}}^q.$$

But $A(m)$ is dominated in (4.2) by a constant independent of m. □

Let q' be the conjugate number of p: $q'=\frac{p}{p-1}$. Taking successively $q=q'$ and $q=2q'$, we see that $\{u_i^m;\ m\ge 1\}$ and $\{(u_i^m)^2;\ m\ge 1\}$ are bounded in $L^{q'}([0,T]\times\mathbb{R}^d)$. We can choose a subsequence such that

$$u_i^m\to v_i \text{ and } (u_i^m)^2\to w_i \text{ weakly in } L^{q'} \text{ as } m\to+\infty.$$

It is easy to check that $v_i(t)$ and $w_i(t)$ solve the transport equation

$$\frac{du}{dt}-Z\cdot\nabla u=0, \tag{4.8}$$

with respectively initial condition ℓ_i and ℓ_i^2.
Now we consider $\beta(s)=(|s|\wedge M)^2$. Applying (3.16), $\beta(v_i)$ satisfies also transport equation (4.8) with the initial condition $\beta(\ell_i)$:

$$\int_{[0,T]\times\mathbb{R}^d}\bigl[-\alpha'(t)F\ (|v_i|\wedge M)^2-\alpha(t)D_Z^*F\ (|v_i|\wedge M)^2\bigr]\,d\gamma dt=\int_{\mathbb{R}^d}\alpha(0)(|\ell_i(x)|\wedge M)^2\,Fd\gamma.$$

Letting $M \to +\infty$, we see that v_i^2 solves the transport equation (4.8) with the initial condition ℓ_i^2. By Theorem 3.4, we have $w_i = v_i^2$. Therefore as $m \to +\infty$, $u_i^m \to v_i$ and $(u_i^m)^2 \to v_i^2$ weakly, from which we deduce that

$$\lim_{m\to+\infty} \int_{[0,T]\times\mathbb{R}^d} \left|u_i^m(t,x) - v_i(t,x)\right|^2 d\gamma dt = 0. \tag{4.9}$$

Now we define

$$X_t(x) = \sum_{i=1}^d v_i(t,x)\varepsilon_i, \tag{4.10}$$

where $\{\varepsilon_1, \cdots, \varepsilon_d\}$ is the canonical basis of $\mathbb{R}^d$. By (4.9),

$$\lim_{m\to+\infty} \int_{[0,T]\times\mathbb{R}^d} |X_t^m(x) - X_t(x)|^2 \, d\gamma dt = 0. \tag{4.11}$$

Proposition 4.3. *For almost all $t \in [0,T]$, the density K_t of $(X_t)_*\gamma$ with respect to γ exists and*

$$\int_{\mathbb{R}^d} K_t |\log K_t| \, d\gamma \le C_T < +\infty, \tag{4.12}$$

where $C_T > 0$ is a constant independent of $t \in [0,T]$.

Proof. By (4.2) and (4.4), there is a constant $C_T > 0$ such that

$$\int_{\mathbb{R}^d} K_t^m \, |\log K_t^m| \, d\gamma \le C_T. \tag{4.13}$$

Then $\sup_{m\ge1} \int_{[0,T]\times\mathbb{R}^d} K_t^m |\log K_t^m| \, d\gamma dt \le TC_T$; therefore up to a subsequence, K^m converges weakly in $L^1([0,T]\times\mathbb{R}^d)$ to a positive function $\tilde{K} \in L^1([0,T]\times\mathbb{R}^d)$ satisfying

$$\int_{[0,T]\times\mathbb{R}^d} \tilde{K}_t \, |\log \tilde{K}_t| \, d\gamma dt \le TC_T.$$

Hence for any $\alpha \in L^\infty([0,T]), \varphi \in C_c(\mathbb{R}^d)$,

$$\int_{[0,T]\times\mathbb{R}^d} \alpha(t)\varphi(x)\tilde{K}_t(x) \, d\gamma dt = \lim_{m\to+\infty} \int_{[0,T]\times\mathbb{R}^d} \alpha(t)\varphi(x)K_t^m(x) \, d\gamma dt. \tag{4.14}$$

On the other hand,

$$\int_{[0,T]\times\mathbb{R}^d} \alpha(t)\varphi(x)K_t^m(x) \, d\gamma dt = \int_0^T \alpha(t)\Big(\int_{\mathbb{R}^d} \varphi(X_t^m) d\gamma\Big) dt. \tag{ii}$$

But by (4.11), again up to a subsequence, X^m converges to X almost everywhere; therefore the right hand side of (ii) converges

$$\int_0^T \alpha(t)\Big(\int_{\mathbb{R}^d} \varphi(X_t) d\gamma\Big) dt.$$

Combining with (i) yields to

$$\int_{[0,T]\times\mathbb{R}^d} \alpha(t)\varphi(x)\tilde{K}_t(x)\, d\gamma dt = \int_0^T \alpha(t)\Big(\int_{\mathbb{R}^d} \varphi(X_t) d\gamma\Big) dt, \quad \text{for all } \alpha \in L^\infty([0,T]).$$

Therefore for $\varphi \in C_c(\mathbb{R}^d)$ given, there exists $L_\varphi \subset [0,T]$ of full measure such that

$$\int_{\mathbb{R}^d} \varphi(X_t) d\gamma = \int_{\mathbb{R}^d} \varphi\, \tilde{K}_t\, d\gamma, \quad \text{for all } t \in L_\varphi. \tag{iii}$$

By separability of $C_c(\mathbb{R}^d)$, there exists $L \subset [0,T]$ of full measure such that for $t \in L$,

$$\int_{\mathbb{R}^d} \varphi(X_t) d\gamma = \int_{\mathbb{R}^d} \varphi\, \tilde{K}_t\, d\gamma, \quad \text{for all } \varphi \in C_c(\mathbb{R}^d).$$

It follows that for $t \in L$, $\tilde{K}_t$ is the density of $(X_t)_*\gamma$ with respect to γ. □

Proposition 4.4. *For any $q > 1$ and almost all $t < \frac{q-1}{q^2}\frac{\lambda_0}{2e}$,*

$$||K_t||_{L^q}^q \le \int_{\mathbb{R}^d} \exp\Big(\frac{q^2 t}{q-1} |\mathrm{div}_\gamma(Z)|\Big)\, d\gamma. \tag{4.15}$$

Proof. By (4.3), for $\frac{q^2 t}{q-1} < \lambda < \frac{\lambda_0}{2e}$,

$$\sup_{m\ge 1} \int_{\mathbb{R}^d} e^{\lambda |\mathrm{div}_\gamma(B_m)|}\, d\gamma < +\infty.$$

Recall that $\mathrm{div}_\gamma(B_m)$ converges to $\mathrm{div}_\gamma(Z)$ in L^p; up to a subsequence,

$$\lim_{m\to+\infty} \int_{\mathbb{R}^d} e^{\frac{q^2 t}{q-1}|\mathrm{div}_\gamma(B_m)|}\, d\gamma = \int_{\mathbb{R}^d} e^{\frac{q^2 t}{q-1}|\mathrm{div}_\gamma(Z)|}\, d\gamma.$$

Combining with (4.3), we get

$$\liminf_{m\to+\infty} ||K_t^m||_{L^q}^q \le \int_{\mathbb{R}^d} e^{\frac{q^2 t}{q-1}|\mathrm{div}_\gamma(Z)|}\, d\gamma. \tag{4.16}$$

Now using (4.3) and (4.11), for almost all $t \in [0, T_0]$, there exists a subsequence m_n such that $K_t^{m_n}$ converges weakly to K_t in $L^q(\mathbb{R}^d, \gamma)$ and $X_t^{m_n}$ converges to X_t almost everywhere; therefore letting $m_n \to +\infty$ in

$$\int_{\mathbb{R}^d} \varphi(X_t^{m_n})\, d\gamma = \int_{\mathbb{R}^d} \varphi\, K_t^{m_n}\, d\gamma, \quad \text{for all } \varphi \in C_b(\mathbb{R}^d)$$

we see that $\int_{\mathbb{R}^d} \varphi(X_t)\, d\gamma = \int_{\mathbb{R}^d} \varphi\, K_t\, d\gamma$ for all $\varphi \in C_b(\mathbb{R}^d)$. This means that K_t is the density of $(X_t)_*\gamma$ with respect to γ. According to (4.16), we have

$$\Big|\int_{\mathbb{R}^d} \varphi\, K_t\, d\gamma\Big| \le ||\varphi||_{L^{q'}} \Big(\int_{\mathbb{R}^d} e^{\frac{q^2 t}{q-1}|\mathrm{div}_\gamma(Z)|}\, d\gamma\Big)^{1/q}, \quad \text{for a.e. } t < \frac{q-1}{q^2}\frac{\lambda_0}{2e}.$$

The result (4.15) holds. □

In what follows, we will take $q = 2$ and fix $T_0 < \frac{\lambda_0}{8e}$ and consider the subsequence such that X^m converges to X almost everywhere.

Proposition 4.5. *Uniformly with respect to* $[0, T_0]$, *as* $m \to +\infty$

$$\int_0^t B_m(X_s^m)\, ds \to \int_0^t Z(X_s)\, ds \quad \textit{in all } L^q(\mathbb{R}^d, \gamma).$$

Proof. For each $t \in [0, T_0]$,

$$\begin{aligned} &\Big|\int_0^t B_m(X_s^m)\, ds - \int_0^t Z(X_s)\, ds\Big| \\ &\le \int_0^t |B_m(X_s^m) - Z(X_s^m)|\, ds + \int_0^t |Z(X_s^m) - Z(X_s)|\, ds. \end{aligned}$$

Let $q > 1$. Then there is a constant $C_{q,T_0} > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^d} \sup_{t \le T_0} \Big|\int_0^t B_m(X_s^m)\, ds - \int_0^t Z(X_s)\, ds\Big|^q d\gamma \\ &\le C_{q,T_0} \Big\{\int_0^{T_0}\int_{\mathbb{R}^d} |B_m(X_s^m) - Z(X_s^m)|^q\, d\gamma ds + \int_0^{T_0}\int_{\mathbb{R}^d} |Z(X_s^m) - Z(X_s)|^q\, d\gamma ds\Big\} \\ &= J_m^1 + J_m^2, \text{ respectively.} \end{aligned}$$

For estimating J_m^1, we use (4.3) for $q = 2$; then there exists a constant C_{λ_0,T_0} such that

$$J_m^1 = \int_0^{T_0}\int_{\mathbb{R}^d} |B_m - Z|^q K_s^m\, d\gamma ds \le C_{\lambda_0,T_0} \cdot \Big(\int_{\mathbb{R}^d} |B_m - Z|^{2q}\, d\gamma\Big)^{1/2} \to 0$$

as $m \to +\infty$. For estimating J_m^2, for any $\varepsilon > 0$, take $V \in C_b(\mathbb{R}^d)$ such that $||Z - V||_{L^{2q}}^q < \varepsilon$. Then

$$\begin{aligned} \int_{\mathbb{R}^d} |Z(X_s^m) - Z(X_s)|^q\, d\gamma \le C_q \Big\{&\int_{\mathbb{R}^d} |Z(X_s^m) - V(X_s^m)|^q\, d\gamma \\ &+ \int_{\mathbb{R}^d} |Z(X_s) - V(X_s)|^q\, d\gamma + \int_{\mathbb{R}^d} |V(X_s^m) - V(X_s)|^q\, d\gamma\Big\}. \end{aligned} \tag{4.17}$$

Using estimates on densities (4.3) and (4.15), there exists a constant $C_{q,\lambda_0,T_0} > 0$ such that

$$\int_0^{T_0}\int_{\mathbb{R}^d} |Z(X_s^m) - V(X_s^m)|^q\, d\gamma ds + \int_0^{T_0}\int_{\mathbb{R}^d} |Z(X_s) - V(X_s)|^q\, d\gamma ds \le 2C_{q,\lambda_0,T_0}\, \varepsilon.$$

Combining with (4.17), we get $\limsup_{m\to+\infty} J_m^2 \le 2C_{q,\lambda_0,T_0}\, \varepsilon$. Letting $\varepsilon \to 0$ completes the proof. □

Now letting $m \to +\infty$ in

$$X_t^m(x) = x + \int_0^t B_m(X_s^m(x))\, ds;$$

it holds in $L^1([0, T_0] \times \mathbb{R}^d)$,

$$X_t(x) = x + \int_0^t Z(X_s(x))\, ds. \tag{4.18}$$

Note that the right hand side of (4.18) is continuous with respect to $t \in [0, T_0]$ for γ-a.e $x \in \mathbb{R}^d$. Now we redefine

$$\tilde{X}_t(x) = x + \int_0^t Z(X_s(x))\, ds, \quad \text{for } t \in [0, T_0].$$

Obviously for $t \in [0, T_0]$:

$$\tilde{X}_t(x) = x + \int_0^t Z(\tilde{X}_s(x))\, ds. \tag{4.19}$$

Proposition 4.6. *For each $t \in [0, T_0]$, the density K_t of $(\tilde{X}_t)_*\gamma$ with respect to γ admits the explicit expression*

$$K_t(x) = \exp\Big(\int_0^t \mathrm{div}_\gamma(Z)(\tilde{X}_{-s})\, ds\Big), \tag{4.20}$$

where $\tilde{X}_{-s}$ solves (4.19) *replacing Z by $-Z$.*

Proof. Proceeding as above, we have first for any $q > 1$,

$$\lim_{m\to+\infty} \int_{\mathbb{R}^d} \Big[\sup_{t\le T_0} \Big| \int_0^t \mathrm{div}_\gamma(B_m)(X^m_{-s})\, ds - \int_0^t \mathrm{div}_\gamma(Z)(\tilde{X}_{-s})\, ds\Big|^q\Big]\, d\gamma = 0.$$

Up to a subsequence, for almost all $x \in \mathbb{R}^d$, for each $t \in [0, T_0]$,

$$K^m_t(x) \to \exp\Big(\int_0^t \mathrm{div}_\gamma(Z)(\tilde{X}_{-s})\, ds\Big), \quad \text{as } m \to +\infty.$$

Now for $\varphi \in C_b(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(\tilde{X}_t)\, d\gamma &= \lim_{m\to+\infty} \int_{\mathbb{R}^d} \varphi(X^m_t)\, d\gamma \\ &= \lim_{m\to+\infty} \int_{\mathbb{R}^d} \varphi K^m_t\, d\gamma = \int_{\mathbb{R}^d} \varphi \, \exp\Big(\int_0^t \mathrm{div}_\gamma(Z)(\tilde{X}_{-s})\, ds\Big)\, d\gamma,\end{aligned}$$

this last passage is guaranteed by the uniform integrability of K^m_t, see (4.13). □

Definition 4.7. *For $t \in [0, T_0]$, we define*

$$\tilde{X}_{t+T_0}(x) = \tilde{X}_t(\tilde{X}_{T_0}(x)).$$

Replacing x by $\tilde{X}_{T_0}(x)$ in (4.19), we have

$$\begin{aligned}\tilde{X}_{t+T_0}(x) &= \tilde{X}_{T_0}(x) + \int_0^t Z(\tilde{X}_s(\tilde{X}_{T_0}(x)))\, ds \\ &= \tilde{X}_{T_0}(x) + \int_0^t Z(\tilde{X}_{s+T_0}(x))\, ds \\ &= \tilde{X}_{T_0}(x) + \int_{T_0}^{t+T_0} Z(\tilde{X}_s(x))\, ds \\ &= x + \int_0^{t+T_0} Z(\tilde{X}_s(x))\, ds.\end{aligned}$$

In such a way, we redefine $\{\tilde{X}_t;\ t \in [0, 2T_0]\}$ which satisfies (4.19) for all $t \in [0, 2T_0]$.

Proposition 4.8. *For any $q > 1$,*

$$\lim_{m\to+\infty} \int_{\mathbb{R}^d} \sup_{0\le t\le T} |X_t^m - \tilde{X}_t|^q \, d\gamma = 0. \tag{4.21}$$

Proof. We have first, for $t \in [0, T_0]$,

$$X_t^m(X_{T_0}^m) - \tilde{X}_t(\tilde{X}_{T_0}) = X_t^m(X_{T_0}^m) - \tilde{X}_t(X_{T_0}^m) + \tilde{X}_t(X_{T_0}^m) - \tilde{X}_t(\tilde{X}_{T_0}).$$

Then there exists a constant $C_q > 0$ such that

$$\begin{aligned} &\sup_{t\le T_0} |X_t^m(X_{T_0}^m) - \tilde{X}_t(\tilde{X}_{T_0})|^q \\ &\le C_q \Big\{ \sup_{t\le T_0} |X_t^m(X_{T_0}^m) - \tilde{X}_t(X_{T_0}^m)|^q + \sup_{t\le T_0} |\tilde{X}_t(X_{T_0}^m) - \tilde{X}_t(\tilde{X}_{T_0})|^q \Big\} \end{aligned} \tag{4.22}$$

We have

$$\begin{aligned} \int_{\mathbb{R}^d} \sup_{t\le T_0} |X_t^m(X_{T_0}^m) - \tilde{X}_t(X_{T_0}^m)|^q \, d\gamma &= \int_{\mathbb{R}^d} \sup_{t\le T_0} |X_t^m - \tilde{X}_t|^q \, K_{T_0}^m \, d\gamma \\ &\le C_{\lambda_0, T_0} \cdot \Big(\int_{\mathbb{R}^d} \sup_{t\le T_0} |X_t^m - \tilde{X}_t|^{2q} \, d\gamma \Big)^{1/2} \to 0 \end{aligned}$$

as $m \to +\infty$, due to Proposition 4.5. For estimating the second term in (4.22), we remark that

$$\tilde{X}_\cdot : \mathbb{R}^d \to C([0, T_0], \mathbb{R}^d)$$

satisfies $\int_{\mathbb{R}^d} \sup_{t\le T_0} |\tilde{X}_t|^{2q} \, d\gamma < +\infty$. Note that the estimate (4.15) holds for T_0. Therefore we can proceed as for estimating J_m^2 in the proof of Proposition 4.5. Finally, we get

$$\lim_{m\to+\infty} \int_{\mathbb{R}^d} \sup_{t\le T_0} |X_t^m(X_{T_0}^m) - \tilde{X}_t(\tilde{X}_{T_0})|^q \, d\gamma = 0;$$

in other words

$$\lim_{m\to+\infty} \int_{\mathbb{R}^d} \sup_{t\le T_0} |X_{t+T_0}^m - \tilde{X}_{t+T_0}|^q \, d\gamma = 0.$$

Combining this with Proposition 4.5, we get

$$\lim_{m\to+\infty} \int_{\mathbb{R}^d} \sup_{t\le 2T_0} |X_t^m - \tilde{X}_t|^q \, d\gamma = 0.$$

Now considering $X_t^m(X_{T_0}^m) - \tilde{X}_t(\tilde{X}_{T_0})$ for $t \in [0, 2T_0]$, we get the result

$$\lim_{m\to+\infty} \int_{\mathbb{R}^d} \sup_{t\le 3T_0} |X_t^m - \tilde{X}_t|^q \, d\gamma = 0.$$

In this way, we obtain finally (4.21). □

Theorem 4.9. *We have for any $t, s \in [-T, T]$ such that $t+s \in [-T, T]$,*

$$\tilde{X}_{t+s} = \tilde{X}_t(\tilde{X}_s). \tag{4.23}$$

Furthermore, $(\tilde{X}_t)_{t\in[-T,T]}$ is the unique flow of maps such that $(\tilde{X}_t)_*\gamma = K_t\gamma$ with K_t given in (4.20) *and solves*

$$\tilde{X}_t(x) = x + \int_0^t Z(\tilde{X}_s)\, ds.$$

Proof. We prove only the uniqueness. Let $(Y_t)_{t\in[-T,T]}$ be another flow of maps enjoying the same properties as for $(\tilde{X}_t)_{t\in[-T,T]}$. Let $\ell : \mathbb{R}^d \to \mathbb{R}$ be a linear map and set $u_t = \ell(Y_t)$. For any $\alpha \in C_c^1([0,T))$ and $\varphi \in C_b^1(\mathbb{R}^d)$, consider

$$\Delta_\eta = \int_{[0,T]\times\mathbb{R}^d} \frac{\alpha(t+\eta)-\alpha(t)}{\eta}\varphi(x)u_t(x)\, d\gamma dt. \tag{i}$$

For $\eta > 0$ small enough, $\int_0^T \alpha(t+\eta)\varphi(x)u_t(x)\, dt = \int_\eta^T \alpha(t)\varphi(x)u_{t-\eta}(x)\, dt$. Then

$$\Delta_\eta = \int_\eta^T\int_{\mathbb{R}^d} \alpha(t)\varphi\, \frac{u_{t-\eta}-u_t}{\eta}\, d\gamma dt - \frac{1}{\eta}\int_0^\eta\Big(\int_{\mathbb{R}^d}\alpha(t)\varphi\, u_t\, d\gamma\Big)dt. \tag{ii}$$

We have

$$\int_{\mathbb{R}^d}\varphi(x)\frac{u_{t-\eta}-u_t}{\eta}\, d\gamma = \int_{\mathbb{R}^d}\frac{\varphi(Y_\eta)K_{-\eta}-\varphi}{\eta}\, u_t\, d\gamma. \tag{iii}$$

Note that

$$\lim_{\eta\to 0}\int_{\mathbb{R}^d}\frac{\varphi(Y_\eta)K_{-\eta}-\varphi}{\eta}\, u_t\, d\gamma = \int_{\mathbb{R}^d}\big(\nabla\varphi\cdot Z - \varphi\ \mathrm{div}_\gamma(Z)\big)\, u_t\, d\gamma.$$

Letting $\eta \to 0$ in $(i)-(iii)$, we get

$$\int_{[0,T]\times\mathbb{R}^d}\big(-\alpha'(t)\varphi(x)u_t(x) - \alpha(t)D_Z^*\varphi(x)u_t(x)\big)\, d\gamma(x)dt = \int_{\mathbb{R}^d}\alpha(0)\varphi(x)u_0(x)\, d\gamma(x).$$

Therefore u_t solves the transport equation (4.6) with initial data ℓ. Now by Theorem 3.4, we get $\ell(Y_t) = \ell(\tilde{X}_t)$ for $t \in [0,T]$ for each linear form. Hence $Y_t = \tilde{X}_t$ for $t \in [0,T]$. □

5 Estimate for the commutator

The main objective of this section is to prove Theorem 3.2. We will do it in the framework of the Wiener space and put forward the connection between the commutator estimate in [3] and the geometric analysis on the Wiener space. Let $W = C_0([0,1],\mathbb{R})$ and H be the Cameron-Martin subspace of W: $H = \{h \in W; \int_0^1 |\frac{dh}{ds}|^2\, ds < +\infty\}$ and μ be the Wiener measure on W. We have for $\ell \in W^*$ (dual of W):

$$\int_W e^{\sqrt{-1}\,\ell(w)}d\mu(w) = e^{-|\ell|_H^2/2},$$

where $|\cdot|_H$ is the norm in H. In the following, we fix an orthonormal basis $\{h_i : i \geq 1\}$ of H, with $h_i \in W^*$ for all $i \geq 1$.

We refer to [11] or to a short book [8] for the background in Malliavin calculus. For a $Z \in L^p(W, K)$, where $p > 1$ and K is a separable Hilbert space, we say that $Z \in \mathbb{D}_1^p(W, K)$ if there exists $\nabla Z \in L^p(W, H \otimes K)$ such that for each $h \in H$,

$$\langle \nabla Z, h \rangle = D_h Z = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Z(w + \varepsilon h) \quad \text{holds in } L^{p-}.$$

The space $\mathbb{D}_1^p(W, K)$ is complete under the norm: $\|Z\|_{1,p}^p = \|Z\|_{L^p}^p + \|\nabla Z\|_{L^p}^p$. A K-valued functional Z is called cylindrical if there are N, $M \geq 1$, $f_i \in C_b^\infty(\mathbb{R}^M)$ and $k_i \in K$ $(1 \leq i \leq N)$, such that

$$Z = \sum_{i=1}^N f_i(h_1(w), \cdots, h_M(w))\, k_i.$$

By the Schmidt orthogonalization procedure, we may always assume that $\{k_1, \cdots, k_N\}$ is a orthonormal family. Note that $Z : W \to K$ is Fréchet differentiable of any order. We denote by $\mathrm{Cylin}(W, K)$ the space of K-valued cylindrical functionals. If $K = \mathbb{R}$, we simply write $\mathbb{D}_1^p(W)$ and $\mathrm{Cylin}(W)$. It is known that K-valued cylindrical functions are dense in $\mathbb{D}_1^p(W, K)$. A basic result in Malliavin calculus is that $\mathrm{div}_\mu(Z) \in L^p(W)$ exists for $Z \in \mathbb{D}_1^p(W, H)$, where the divergence $\mathrm{div}_\mu(Z)$ is defined by $\int_W F \mathrm{div}_\mu(Z)\, d\mu = \int_W \langle \nabla F, Z \rangle_H \, d\mu$ and there exists $C_p > 0$, such that

$$\|\mathrm{div}_\mu(Z)\|_{L^p} \leq C_p \, \|Z\|_{\mathbb{D}_1^p}. \tag{5.1}$$

The Ornstein-Uhlenbeck semigroup P_ε on W is defined by the Mehler formula:

$$P_\varepsilon F(x) = \int_X F\big(e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon}} y\big)\, d\mu(y). \tag{5.2}$$

Here are some basic properties of the semigroup P_ε:

Proposition 5.1. *(1) For any $\varepsilon > 0$, $P_\varepsilon(\mathrm{Cylin}(W)) \subset \mathrm{Cylin}(W)$.*
(2) For any $\varepsilon > 0$ and $p \in [1, +\infty)$, we have for all $u \in L^p(W)$, $\|P_\varepsilon u\|_{L^p} \leq \|u\|_{L^p}$ and $\lim_{\varepsilon \to 0} \|P_\varepsilon u - u\|_{L^p} = 0$.
(3) P_ε is self-adjoint in $L^2(W)$. Furthermore, for any $p \in]1, +\infty[$ and $u \in L^p(W)$, $v \in L^{p'}(W)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\int_W u P_\varepsilon v \, d\mu = \int_W v P_\varepsilon u \, d\mu.$$

(4) For every $\varepsilon > 0$ and $p > 1$, we have $P_\varepsilon u \in \mathbb{D}_1^p(W)$ for any $u \in L^p(W)$, and there is $C_{p,\varepsilon} > 0$ such that

$$\|\nabla P_\varepsilon u\|_{L^p(W,H)} \leq C_{p,\varepsilon} \|u\|_{L^p(W)}.$$

Notice that this last result was due to H. Sugita [13]. Given a cylindrical vector field $Z : W \to H$:

$$Z(w) = \sum_{i=1}^N f_i(h_1(w), \cdots, h_M(w)) h_i \in W^*,$$

we consider the quantity, for $x, y \in W$,

$$A_Z(x, y) = \langle Z'(x) \cdot y, y \rangle - \sum_{i=1}^N (D_{h_i} f_i)(x)$$

$$= \sum_{i=1}^N \langle \nabla f_i(x), y\rangle \langle h_i, y\rangle - \sum_{i=1}^N (D_{h_i} f_i)(x), \tag{5.3}$$

where $Z'(x) : W \to W^*$ denotes the Fréchet differential of $x \to Z(x)$.

Proposition 5.2. *Define* $(\nabla Z(x))^*$ *by* $\big\langle (\nabla Z(x))^*, h_1 \otimes h_2 \big\rangle_{H\otimes H} = \big\langle \nabla Z(x), h_2 \otimes h_1 \big\rangle_{H\otimes H}$. *Then*

$$\int_W |A_Z(x,y)|^2 d\mu(y) = |\nabla Z(x)|^2_{H\otimes H} + \langle \nabla Z(x), (\nabla Z(x))^* \rangle_{H\otimes H}. \tag{5.4}$$

Proof. Using the second expression in (5.3),

$$\begin{aligned} |A_Z(x,y)|^2 \;=\; & \sum_{i,j=1}^N \langle \nabla f_i(x), y\rangle \langle \nabla f_j(x), y\rangle \langle h_i, y\rangle \langle h_j, y\rangle \\ & -2\Big(\sum_{i=1}^N \langle \nabla f_i(x), y\rangle \langle h_i, y\rangle \Big) \Big(\sum_{i=1}^N (D_{h_i} f_i)(x) \Big) + \Big(\sum_{i=1}^N (D_{h_i} f_i)(x) \Big)^2. \end{aligned}$$

We obtain by the integration by parts formula,

$$\begin{aligned} & \sum_{i,j=1}^N \int_W \langle \nabla f_i(x), y\rangle \langle \nabla f_j(x), y\rangle \langle h_i, y\rangle \langle h_j, y\rangle d\mu(y) \\ = & \sum_{i,j=1}^N \int_W D_{h_j} \big[\langle \nabla f_i(x), y\rangle \langle \nabla f_j(x), y\rangle \langle h_i, y\rangle \big] d\mu(y), \end{aligned}$$

which is equal to

$$\begin{aligned} \sum_{i,j=1}^N \Big(& \int_W \langle \nabla f_i(x), h_j\rangle \langle \nabla f_j(x), y\rangle \langle h_i, y\rangle d\mu(y) + \int_W \langle \nabla f_i(x), y\rangle \langle \nabla f_j(x), h_j\rangle \langle h_i, y\rangle d\mu(y) \\ & + \int_W \langle \nabla f_i(x), y\rangle \langle \nabla f_j(x), y\rangle \langle h_i, h_j\rangle d\mu(y) \Big). \end{aligned} \tag{5.5}$$

Again by integrating by parts,

$$\begin{aligned} \int_W \langle \nabla f_i(x), h_j\rangle \langle \nabla f_j(x), y\rangle \langle h_i, y\rangle d\mu(y) \;&=\; \langle \nabla f_i(x), h_j\rangle \int_W D_{h_i} \langle \nabla f_j(x), y\rangle d\mu(y) \\ &=\; D_{h_j} f_i(x)\, D_{h_i} f_j(x). \end{aligned} \tag{5.6}$$

Similarly,

$$\int_W \langle \nabla f_i(x), y\rangle \langle \nabla f_j(x), h_j\rangle \langle h_i, y\rangle d\mu(y) = D_{h_j} f_j(x)\, D_{h_i} f_i(x). \tag{5.7}$$

The equalities (5.5), (5.6) and (5.7) lead to

$$\begin{aligned} & \sum_{i,j=1}^N \int_W \langle \nabla f_i(x), y\rangle \langle \nabla f_j(x), y\rangle \langle h_i, y\rangle \langle h_j, y\rangle d\mu(y) \\ = & \sum_{i,j=1}^N D_{h_j} f_i(x)\, D_{h_i} f_j(x) + \Big(\sum_{i=1}^N D_{h_i} f_i(x) \Big)^2 + \sum_{i=1}^N |\nabla f_i(x)|^2_H, \end{aligned}$$

since

$$\int_W \langle \nabla f_i(x), y\rangle \langle \nabla f_j(x), y\rangle \langle h_i, h_j\rangle d\mu(y) = \delta_{ij} \int_W \langle \nabla f_i(x), y\rangle^2 d\mu(y) = \delta_{ij} |\nabla f_i(x)|_H^2.$$

In the same way, we have

$$\int_W \langle \nabla f_i(x), y\rangle \langle h_i, y\rangle d\mu(y) = \langle \nabla f_i(x), h_i\rangle = D_{h_i} f_i(x).$$

Therefore

$$\int_W |A_Z(x,y)|^2 d\mu(y) = \sum_{i,j=1}^N D_{h_j} f_i(x)\, D_{h_i} f_j(x) + \sum_{i=1}^N |\nabla f_i(x)|_H^2. \tag{5.8}$$

Now $\nabla Z(x) = \sum_{i=1}^N \nabla f_i(x) \otimes h_i$, we have

$$|\nabla Z(x)|_{H\otimes H}^2 = \sum_{i=1}^N |\nabla f_i(x)|_H^2$$

and

$$\langle \nabla Z(x), (\nabla Z(x))^*\rangle_{H\otimes H} = \sum_{i,j=1}^N D_{h_j} f_i(x) D_{h_i} f_j(x).$$

So we get (5.4). □

Proposition 5.3. *Denote by* $I(x,y) = \langle Z'(x)\cdot y, y\rangle - \langle Z(x), x\rangle$, *then for any* $1 < p \le 2$,

$$\|I\|_{L^p(W\times W)} \le C_p \|Z\|_{\mathbb{D}_1^p(W,H)}. \tag{5.9}$$

Proof. We deduce from Proposition 5.2 that

$$\|A_Z(x,\cdot)\|_{L^2(W)} \le \sqrt{2}\, |\nabla Z(x)|_{H\otimes H}. \tag{5.10}$$

Note that $\mathrm{div}_\mu(Z) = \sum_i f_i(x)\langle h_i, x\rangle - \sum_i D_{h_i} f_i(x)$. Then

$$\begin{aligned} I(x,y) &= \langle Z'(x)\cdot y, y\rangle - \sum_{i=1}^N D_{h_i} f_i(x) + \sum_{i=1}^N D_{h_i} f_i(x) - \sum_{i=1}^N f_i(x)\langle h_i, x\rangle \\ &= A_Z(x,y) - \mathrm{div}_\mu(Z)(x), \end{aligned}$$

and

$$\|I\|_{L^p(W\times W)} \le \|A_Z\|_{L^p(W\times W)} + \|\mathrm{div}_\mu(Z)\|_{L^p(W)}. \tag{5.11}$$

For any $1 < p \le 2$, we have

$$\|A_Z\|_{L^p(W\times W)} = \Big(\int_W \|A_Z(x,\cdot)\|_{L^p(W)}^p d\mu(x)\Big)^{\frac{1}{p}} \le \Big(\int_W \|A_Z(x,\cdot)\|_{L^2(W)}^p d\mu(x)\Big)^{\frac{1}{p}}.$$

This plus (5.10) gives

$$\|A_Z\|_{L^p(W\times W)} \le \Big(\int_W \big(\sqrt{2}\, |\nabla Z(x)|_{H\otimes H}\big)^p d\mu(x)\Big)^{\frac{1}{p}} \le \sqrt{2}\, \|Z\|_{\mathbb{D}_1^p(W,H)}.$$

Now by (5.1) and (5.11), we know that there exists $C_p > 0$ such that

$$\|I\|_{L^p(W\times W)} \le C_p \|Z\|_{\mathbb{D}_1^p(W,H)},$$

which completes the proof. □

Remark 5.4. *For $p = 2$, the fillowing equality holds*

$$\int_{X\times X} |I(x,y)|^2 d\mu(x)d\mu(y) = \int_X |\nabla Z|^2_{H\otimes H} d\mu + 2\int_X |\mathrm{div}_\mu(Z)|^2\, d\mu - \int_X |Z|_H^2\, d\mu. \tag{5.12}$$

In fact it is obvious that $\int_X A_Z(x,y)\, d\mu(y) = 0$. Then

$$\int_{X\times X} |I(x,y)|^2 d\mu(x)d\mu(y) = \int_{X\times X} |A_Z(x,y)|^2\, d\mu(x)d\mu(y) + \int_X |\mathrm{div}_\mu(Z)|^2 d\mu.$$

But according to (5.4),

$$\int_{X\times X} |A_Z(x,y)|^2 d\mu(x)d\mu(y) = \int_X |\nabla Z|^2_{H\otimes H}\, d\mu + \int_X \big\langle \nabla Z, (\nabla Z)^*\big\rangle_{H\otimes H}\, d\mu.$$

A version of the Weitzenböck formula on the Wiener space (see [11], [8]) reads as

$$\int_X |\mathrm{div}_\mu(Z)|^2 d\mu = \int_X |Z|^2_H\, d\mu + \int_X \big\langle \nabla Z, (\nabla Z)^*\big\rangle_{H\otimes H}\, d\mu.$$

The result (5.12) follows. □

Proposition 5.5. *Set $\tilde Z(x,y) = \langle Z(x), y\rangle$ and*

$$O_\varepsilon(x,y) = \big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y, -\sqrt{1-e^{-2\varepsilon}}\, x + e^{-\varepsilon}y\big).$$

Then

$$P_\varepsilon(\mathrm{div}_\mu(Z))(x) = -\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\int_W \tilde Z(O_\varepsilon(x,y))\, d\mu(y). \tag{5.13}$$

Proof. We have

$$\begin{aligned} P_\varepsilon(D_{h_i} f_i)(x) &= \int_W D_{h_i} f_i\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y\big) d\mu(y) \\ &= \frac{1}{\sqrt{1-e^{-2\varepsilon}}}\int_W f_i\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y\big)\langle h_i, y\rangle d\mu(y) \end{aligned}$$

and

$$P_\varepsilon(f_i\langle h_i,\cdot\rangle)(x) = \int_W f_i\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y\big)\langle h_i, e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y\rangle d\mu(y).$$

Therefore

$$\begin{aligned} &P_\varepsilon(f_i\langle h_i,\cdot\rangle - D_{h_i} f_i)(x) \\ &= -\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\int_W f_i\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y\big)\langle h_i, -\sqrt{1-e^{-2\varepsilon}}\, x + e^{-\varepsilon}y\rangle d\mu(y), \end{aligned} \tag{5.14}$$

since

$$e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y-\frac{1}{\sqrt{1-e^{-2\varepsilon}}}y=-\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\big(-\sqrt{1-e^{-2\varepsilon}}\,x+e^{-\varepsilon}y\big).$$

Summing up both sides of (5.14), we get the result. □

Let $R_{-\pi/2}: W\times W\to W\times W$ be the rotation defined by $(x,y)\to(y,-x)$. Using the notation $\tilde{Z}$, the term $I(x,y)$ in Proposition (5.9) can be expressed by

$$I=\tilde{Z}'(R_{-\pi/2}),$$

where the prime denotes the Fréchet differential.

Theorem 5.6. *Let $v\in \mathrm{Cylin}(W)$ and define $B_\varepsilon(v,Z)=\langle Z,\nabla P_\varepsilon v\rangle-P_\varepsilon(\langle\nabla v,Z\rangle)$. Then for any $p,q,r\geq 1$ with $1<p\leq 2$ and $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, we have*

$$\|B_\varepsilon(v,Z)\|_{L^r}\leq C_p\|v\|_{L^q}\|Z\|_{\mathbb{D}_1^p}. \tag{5.15}$$

Proof. We have

$$\langle Z,\nabla P_\varepsilon v\rangle=\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\int_W v\big(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y\big)\tilde{Z}(x,y)\,d\mu(y),$$

where $\tilde{Z}$ is defined in Proposition 5.5. Replacing Z by vZ in (5.13), we obtain

$$P_\varepsilon(\mathrm{div}_\mu(vZ))=-\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\int_W v\big(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y\big)\tilde{Z}(O_\varepsilon(x,y))\,d\mu(y).$$

Note that $\mathrm{div}_\mu(vZ)=v\mathrm{div}_\mu(Z)-\langle\nabla v,Z\rangle$, then

$$B_\varepsilon(v,Z)=\langle Z,\nabla P_\varepsilon v\rangle+P_\varepsilon(\mathrm{div}_\mu(vZ))-P_\varepsilon(v\mathrm{div}_\mu(Z)). \tag{5.16}$$

The delicate term is

$$\begin{aligned}B_\varepsilon^1 &:= \langle Z,\nabla P_\varepsilon v\rangle+P_\varepsilon(\mathrm{div}_\mu(vZ))\\ &= \frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\int_W v\big(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y\big)\big(\tilde{Z}(x,y)-\tilde{Z}(O_\varepsilon(x,y))\big)d\mu(y).\end{aligned}$$

Note that $\frac{d}{d\varepsilon}O_\varepsilon=\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}R_{-\pi/2}\circ O_\varepsilon$. Setting $\tilde{v}(x,y)=v(x)$, we can write B_ε^1 in the form

$$B_\varepsilon^1=-\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\int_W \tilde{v}(O_\varepsilon(x,y))\bigg(\int_0^1\frac{\varepsilon e^{-\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}}I(O_{\varepsilon s}(x,y))ds\bigg)d\mu(y). \tag{5.17}$$

It follows that

$$|B_\varepsilon^1(x)|^r\leq\bigg(\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\bigg)^r\int_W|\tilde{v}(O_\varepsilon(x,y))|^r\cdot\bigg|\int_0^1\frac{\varepsilon e^{-\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}}I(O_{\varepsilon s}(x,y))ds\bigg|^r d\mu(y),$$

therefore

$$\|B_\varepsilon^1\|_{L^r}\leq\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\|\tilde{v}(O_\varepsilon(\cdot,\cdot))\|_{L^q(W\times W)}\bigg\|\int_0^1\frac{\varepsilon e^{-\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}}I(O_{\varepsilon s}(\cdot,\cdot))ds\bigg\|_{L^p(W\times W)}$$

$$\leq \|v\|_{L^q(W)}\|I\|_{L^p(W\times W)},$$

since, by the invariance of $\mu\otimes\mu$ under $O_{\varepsilon s}$,

$$\|I(O_{\varepsilon s}(\cdot,\cdot))\|_{L^p(W\times W)} = \|I\|_{L^p(W\times W)}$$

and

$$\int_0^1 \frac{\varepsilon e^{-\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}}ds \leq e^{\varepsilon}\int_0^1 \frac{\varepsilon e^{-2\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}}ds = e^{\varepsilon}\sqrt{1-e^{-2\varepsilon}}.$$

Combining with (5.16) and by Proposition 5.1 (2), we get

$$\|B_\varepsilon(v,Z)\|_{L^r} \leq \|v\|_{L^q}\big(\|I\|_{L^p(W\times W)} + \|\mathrm{div}_\mu(Z)\|_{L^p}\big).$$

Now the inequality (5.1) and Proposition 5.3 lead to the result. □

By the expression (5.17), for cylindrical v and Z, we have

$$\lim_{\varepsilon\to 0} B_\varepsilon^1(x) = -\frac{1}{\sqrt{2}}\int_W \tilde v(x,y)\bigg(\int_0^1 \frac{1}{\sqrt{2s}}I(x,y)\,ds\bigg)d\mu(y) = -v(x)\int_W I(x,y)d\mu(y).$$

Since

$$\int_W I(x,y)d\mu(y) = \sum_{i=1}^N \big(D_{h_i}f_i(x) - f_i(x)h_i(x)\big) = -\mathrm{div}_\mu(Z)(x),$$

therefore

$$\lim_{\varepsilon\to 0} B_\varepsilon^1(x) = v\mathrm{div}_\mu(Z)(x).$$

Again by (5.16), we have $\lim_{\varepsilon\to 0} B_\varepsilon(v,Z)(x) = 0$. Hence the dominated convergence theorem gives

$$\lim_{\varepsilon\to 0}\|B_\varepsilon(v,Z)\|_{L^r} = 0.$$

Theorem 5.7. *Assume $p,q,r\geq 1$ with $1<p\leq 2$ and $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. Then for any $v\in L^q(W)$ and $Z\in\mathbb{D}_1^p(W,H)$, we have $B_\varepsilon(v,Z)\in L^r(W)$ and*

$$\|B_\varepsilon(v,Z)\|_{L^r} \leq C_p\|v\|_{L^q}\|Z\|_{\mathbb{D}_1^p}. \tag{5.18}$$

Moreover,

$$\lim_{\varepsilon\to 0}\|B_\varepsilon(v,Z)\|_{L^r} = 0. \tag{5.19}$$

Proof. First we fix some $Z\in \mathrm{Cylin}(W,H)$. For any $v\in L^q(W)$, there exists a sequence $\{v_n : n\geq 1\}\subset \mathrm{Cylin}(W)$ such that $\lim_{n\to+\infty}\|v_n - v\|_{L^q} = 0$. By the linearity of $v\mapsto B_\varepsilon(v,Z)$ and Theorem 5.6, we know that $\{B_\varepsilon(v_n,Z) : n\geq 1\}$ is a Cauchy sequence in $L^r(W)$. We denote by $B_\varepsilon(v,Z)$ its limit in $L^r(W)$, and we have

$$\|B_\varepsilon(v,Z)\|_{L^r} = \lim_{n\to+\infty}\|B_\varepsilon(v_n,Z)\|_{L^r} \leq \liminf_{n\to+\infty} C_p\|v_n\|_{L^q}\|Z\|_{\mathbb{D}_1^p} = C_p\|v\|_{L^q}\|Z\|_{\mathbb{D}_1^p},$$

hence (5.18) follows. On the other hand, for every $n\geq 1$,

$$\begin{aligned}\|B_\varepsilon(v,Z)\|_{L^r} &\leq \|B_\varepsilon(v_n,Z)\|_{L^r} + \|B_\varepsilon(v-v_n,Z)\|_{L^r}\\ &\leq \|B_\varepsilon(v_n,Z)\|_{L^r} + C_p\|v-v_n\|_{L^q}\|Z\|_{\mathbb{D}_1^p}.\end{aligned}$$

Letting $\varepsilon \to 0$ and by the above discussion, we get

$$\limsup_{\varepsilon\to 0} \|B_\varepsilon(v,Z)\|_{L^r} \le C_p \|v-v_n\|_{L^q}\|Z\|_{\mathbb{D}_1^p}.$$

Now we conclude (5.19) in this case by letting $n \to +\infty$.
For general $Z \in \mathbb{D}_1^p(W,H)$, using the linearity of $Z \mapsto B_\varepsilon(v,Z)$ and the analogous argument works. □

Let V_n be the subspace of H spanned by $\{h_1, \cdots, h_n\}$ and π_n the orthogonal projection from H to V_n. π_n can be extended to W, and $\gamma_n := (\pi_n)_*\mu$ is the standard Gaussian measure on V_n. Let $\mathbb{E}^{V_n}$ be the conditional expectation with respect to the σ-field generated by cylindrical functions of the form $F = f \circ \pi_n$. For $Z \in \mathbb{D}_1^p(W,H)$ and any $n \ge 1$, there exists $Z_n : V_n \to V_n$ satisfying

$$\mathbb{E}^{V_n}(\pi_n(Z)) = Z_n \circ \pi_n \tag{5.20}$$

and

$$\mathrm{div}_{\gamma_n}(Z_n) \circ \pi_n = \mathbb{E}^{V_n}(\mathrm{div}_\mu(Z)). \tag{5.21}$$

It is well known that

$$\lim_{n\to+\infty} \|Z_n \circ \pi_n - Z\|_{\mathbb{D}_1^p(W,H)} = 0. \tag{5.22}$$

By reproducing the arguments developed in section 3 and 4 via finite dimensional approximation, we obtained in [9] the following result

Theorem 5.8. *Let $Z \in \mathbb{D}_1^p(W,H)$ with $1 < p \le 2$. Assume that*

$$\int_W e^{\lambda_0(|Z|_H + |\mathrm{div}_\mu(Z)|)}\, d\mu < +\infty \quad \textit{for a small } \lambda_0 > 0.$$

Then there is a unique flow of maps $U_t : W \to W$ such that $K_t = d(U_t)_*\mu/d\mu$ admits the expression

$$K_t(x) = \exp\Big(\int_0^t \mathrm{div}_\mu(U_{-s}(x))ds\Big),$$

and for μ-a.s,

$$U_t(x) = x + \int_0^t Z(U_s(x))\, ds.$$

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