ODE, SDE and PDE

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0 Introduction

A.1. Consider the ordinary differential equation (abbreviated as ODE) on $\mathbb{R}^d$

$$\frac{dX_t}{dt} = V(X_t), \quad X_0 = x. \tag{0.1}$$

When the coefficient V is smooth and with the bounded derivative, then the ODE (0.1) can be solved by Picard iteration. Let $X_t(x)$ be the solution to (0.1) with the initial condition x. Then $X_t : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$ and enjoys the property of flows: $X_{t+s} = X_t \circ X_s$.

Consider now the following transport equation

$$\frac{du_t}{dt} + V \cdot \nabla u_t = 0, \quad u_0 = \theta \in C^1(\mathbb{R}^d). \tag{0.2}$$

This is a first order Partial Differential Equation. The solution $(u_t)_{t\geq 0}$ to (0.2) can be expressed in term of X_t, namely

$$u_t = \theta\big(X_t^{-1}\big).$$

A.2. When V satisfies the following Osgood condition

$$|V(x) - V(y)| \leq C|x-y| \log \frac{1}{|x-y|}, \quad |x-y| \leq \delta_0, \tag{0.3}$$

the ODE (0.1) defines a flow of homeomorphisms of $\mathbb{R}^d$. If V admits the divergence, then $u_t = \theta\big(X_t^{-1}\big)$ for $\theta \in C(\mathbb{R}^d)$ satisfies again the transport equation (0.2), but in the following sense:

$$(u_t, \varphi)_{L^2} = (\theta, \varphi)_{L^2} - \int_0^t (u_s, \mathrm{div}(\varphi V))_{L^2}\, ds, \quad \varphi \in C_c^\infty(\mathbb{R}^d),$$

where $(\ ,\ )_{L^2}$ denotes the inner product in $L^2(\mathbb{R}^d)$ with respect to the Lebesgue measure λ_d. An immediate consequence of this consideration is that, whenever $\mathrm{div}(V) = 0$, the flow X_t preserves the Lebesgue measure.

A.3. Due to the linearity, the transport equation (0.2) can be uniquely solved under weak conditions on V; namely under the condition

$$V \in W^{1,q} \quad \text{and} \quad \mathrm{div}(V) \in L^\infty, \tag{0.4}$$

using the argument of smoothing. In the opposite direction, the transport equation (0.2) allows to solve the ODE (0.1) in order to get a flow of quasi-invariant measurable maps. To this end, we shall follow the method developed by L. Ambrosio, which consists of three steps:

(i) Consider the mass conservation equation

$$\frac{\partial \mu_t}{\partial t} - D_x \cdot (V\mu_t) = 0, \quad \mu_0 \text{ given.} \tag{0.5}$$

(ii) Construct a coupling measure η on the product space $\mathbb{R}^d \times W$, where $W = C([0,T],\mathbb{R}^d)$ in the sense that

$$(\pi_{\mathbb{R}^d})_*\eta = \mu_0, \quad (e_t)_*\eta = \mu_t, \tag{0.6}$$

where $e_t : (x,\gamma) \to \gamma(t)$, $\pi_{\mathbb{R}^d}(x,\gamma) = x$, such that

$$\gamma(t) = x + \int_0^t V(\gamma(s))ds, \quad \text{holds} \quad \eta\text{-a.s.} \tag{0.7}$$

(iii) Show that η is supported by a graph: there exists $X : \mathbb{R}^d \to W$ such that

$$\eta = (I \times X)_*\mu_0.$$

Then $(X_t(x))$ solves the ODE (0.1).

B.1. Consider the Itô stochastic differential equation (abbreviated as SDE)

$$dX_t = \sigma(X_t)dw_t + V(X_t)dt \tag{0.8}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$. When the coefficients are globally Lipschitz, then (0.8) defines a stochastic flow of homeomorphisms on $\mathbb{R}^d$: there is a full measure subset $\Omega_o \subset \Omega$ such that for $w \in \Omega_o$,

$$X_t(w,\cdot) : \mathbb{R}^d \to \mathbb{R}^d \quad \text{is a homeomorphism.}$$

Under the stronger conditions $\sigma \in C_b^2$ and $V \in C_b^1$, $(X_t)_{t\geq 0}$ is now a flow of diffeomorphisms of $\mathbb{R}^d$.

Let P_x be the law of $w \to X_\cdot(w,x)$ on $C([0,T],\mathbb{R}^d)$. For a given probability measure μ_0, consider $P_{\mu_0} = \int_{\mathbb{R}^d} P_x d\mu_0(x)$. Let $e_t : C([0,T],\mathbb{R}^d) \to \mathbb{R}^d$ such that $e_t(\gamma) = \gamma(t)$. Then $\mu_t := (e_t)_* P_{\mu_0}$ solves the Fokker-Planck equation

$$\frac{\partial \mu_t}{\partial t} = L^*\mu_t, \quad \mu_{t|_{t=0}} = \mu_0, \tag{0.9}$$

where L^* is the formal dual operator of

$$L = \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d V_i \frac{\partial}{\partial x_i}, \quad a = \sigma\sigma^*. \tag{0.10}$$

B.2. The operator L is related to the weak solutions of SDE (0.8). A better knowledge, for example elliptic estimates, on partial differential equations (abbreviated as PDE) makes possible to prove that the weak solution is indeed the strong one in some situation.

B.3. In the opposite direction, there are some recent works (see [5], [9]) on SDE (0.8) from the Fokker-Planck equation (0.9) for few regular coefficients.

1 Flow of Homeomorphisms under Osgood Conditions

1.1 Classical Case

Let $(V_t)_{t\in[0,T]}$ be a time-dependent vector field on $\mathbb{R}^d$. Suppose that

$$|V_t(x) - V_t(y)| \le C(t)\,|x-y| \quad \text{for } x, y \in \mathbb{R}^d \tag{1.1.1}$$

with

$$\int_0^T |V_s(x)|\,ds < +\infty, \quad \int_0^T C(s)\,ds < +\infty. \tag{1.1.2}$$

The differential equation

$$\frac{dX_t}{dt} = V(X_t), \quad X_0 = x \tag{1.1.3}$$

can be solved by Picard iteration: define $X_1(t) = x + \int_0^t V_s(x)\,ds$ and for $n \ge 1$

$$X_{n+1}(t) = x + \int_0^t V_s(X_n(s))\,ds.$$

Due to the condition (1.1.2), we see that $t \to X_n(t)$ is absolutely continuous for $n \ge 1$. We have

$$|X_{n+1}(t) - X_n(t)| \le \int_0^t C(s)\,|X_n(s) - X_{n-1}(s)|\,ds.$$

By induction, we prove that there exists a constant $C > 0$ such that

$$|X_{n+1}(t) - X_n(t)| \le C\,\frac{F(t)^n}{n!}, \quad \text{where } F(t) = \int_0^t C(s)\,ds.$$

It follows that $X_n(t)$ converges to X_t, solution to (1.1.3), uniformly on $t \in [0,T]$. Note that $t \to X_t$ is absolutely continuous. Denote now by $X_t(x)$ the solution to (1.1.3) with the initial condition x. Let $t_0 \in [0,T]$ and $Y_t(x)$ be the solution to the ODE:

$$\frac{dY_t}{dt} = -V_{t_0-t}(Y_t), \quad Y_0 = x.$$

We check that $X_{t_0-t}(x)$ and $Y_t(X_{t_0}(x))$ satisfy the same ODE with the initial condition $X_{t_0}(x)$; therefore by uniqueness of solutions, $X_{t_0-t} = Y_t(X_{t_0})$ for $0 \le t \le t_0$. In the same way, $Y_{t_0-t} = X_t(Y_{t_0})$. Now putting $t = t_0$ gives that $X_{t_0}(Y_{t_0}) = Y_{t_0}(X_{t_0}) = I$. Hence

$$X_{t_0}^{-1} = Y_{t_0}.$$

It follows that $x \to X_t(x)$ is a homeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Moreover if V_t is supposed to be C^1 with bounded derivative, then $x \to X_t(x)$ is a diffeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$. To see the derivability of $t \to X_t^{-1}$, it is convenient to consider the ODE

$$\frac{d}{dt}X(t,s,x) = V_t(X(t,s,x)), \quad X(s,s,x) = x.$$

Then $X_t(x) = X(t,0,x)$ and $(X_t^{-1})(x) = X(0,t,x)$.

Now let $\theta \in C^1(\mathbb{R}^d)$. We denote by θ' the differential of θ, that is, for $x \in \mathbb{R}^d$, $\theta'(x)$ is a linear map from $\mathbb{R}^d$ to $\mathbb{R}$; we denote by $\nabla\theta$ the gradient of θ, that is a vector field on $\mathbb{R}^d$ such that $\big\langle \nabla\theta(x), v\big\rangle = \theta'(x)\cdot v$ for $v \in \mathbb{R}^d$. For a vector field V on $\mathbb{R}^d$ and an application $F : \mathbb{R}^d \to \mathbb{R}^m$, we denote also by $D_V F$ the directional derivative of F along V, that is, $(D_V F)(x) = \frac{d}{dt}_{|t=0} F(X_t)$, where X_t is the flow associated to V. Let $u_t = \theta(X_t^{-1})$. Then

$$\frac{du_t}{dt} = \theta'(X_t^{-1}) \cdot \frac{dX_t^{-1}}{dt}. \tag{i}$$

On the other hand,

$$\langle \nabla u_t, V_t\rangle = \theta'(X_t^{-1}) \cdot D_{V_t} X_t^{-1}.$$

Now differentiating the equality $x = X_t^{-1}(X_t(x))$ with respect to the time t, we have

$$0 = \frac{dX_t^{-1}}{dt}(X_t) + (X_t^{-1})'(X_t)\frac{dX_t}{dt} = \frac{dX_t^{-1}}{dt}(X_t) + (D_{V_t}X_t^{-1})(X_t).$$

Since X_t is bijective, so for each $x \in \mathbb{R}^d$, the above equality gives

$$\frac{dX_t^{-1}}{dt} + D_{V_t}X_t^{-1} = 0.$$

According to (i):

$$\frac{du_t}{dt} + V_t \cdot \nabla u_t = 0, \quad u_0 = \theta \in C^1 \tag{1.1.4}$$

here we used $\cdot$ for the inner product. Conversely, if $u_t \in C^1$ is a solution of (1.1.4), then

$$\frac{d}{dt}[u_t(X_t)] = \frac{du_t}{dt}(X_t) + u_t'(X_t)\frac{dX_t}{dt} = \frac{du_t}{dt}(X_t) + (V_t \cdot \nabla u_t)(X_t) = 0,$$

so that $u_t(X_t) = \theta$ or $u_t = \theta(X_t^{-1})$: it is the unique solution to (1.1.4).

Remark 1.1 *The equation* (1.1.3) *is non linear while the equation* (1.1.4) *is linear.* (1.1.3) $\Rightarrow$ (1.1.4)*.*

1.2 Osgood Condition

In this subsection, we suppose that $V : \mathbb{R}^d \to \mathbb{R}^d$ is a time-independent bounded continuous vector field such that

$$|V(x) - V(y)| \le C|x-y| \log\frac{1}{|x-y|}, \quad |x-y| \le \delta < 1. \tag{1.2.1}$$

Then the differential equation

$$\frac{dX_t}{dt} = V(X_t), \quad X_0 = x \tag{1.2.2}$$

admits at least one solution $(X_t)_{t\ge 0}$.

Proposition 1.2 *Under* (1.2.1)*, the differential equation* (1.2.2) *admits a unique solution.*

Proof. Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two solutions to (1.2.2) starting from the same point. Set $\eta_t = X_t - Y_t$ and $\xi_t = |\eta_t|^2$. Let $\varepsilon > 0$ be a parameter and define the function

$$\Psi_\varepsilon(\xi) = \int_0^\xi \frac{ds}{s\log\frac{1}{s} + \varepsilon}, \quad 0 \leq \xi \leq \delta,$$

and $\Phi_\varepsilon = e^{\Psi_\varepsilon}$. Then $\Phi_\varepsilon' = \frac{\Phi_\varepsilon}{\xi\log\frac{1}{\xi}+\varepsilon}$. Define

$$\tau = \inf\{t > 0 : \xi_t \geq \delta^2\}.$$

Then by (1.2.1), for $t \leq \tau$,

$$|\langle \eta_t, V(X_t) - V(Y_t)\rangle| \leq |\eta_t| \cdot C|\eta_t| \log\frac{1}{|\eta_t|} = \frac{C}{2}\xi_t \log\frac{1}{\xi_t}.$$

Hence

$$\begin{aligned}\left|\frac{d}{dt}\Phi_\varepsilon(\xi_t)\right| &= \left|\Phi_\varepsilon'(\xi_t)\frac{d\xi_t}{dt}\right| = \left|\Phi_\varepsilon'(\xi_t)\cdot 2\langle \eta_t, \frac{d\eta_t}{dt}\rangle\right| \\ &\leq \Phi_\varepsilon(\xi_t)\cdot\frac{1}{\xi_t\log\frac{1}{\xi_t}+\varepsilon}\cdot C\xi_t\log\frac{1}{\xi_t} \leq C\Phi_\varepsilon(\xi_t).\end{aligned}$$

This and $\xi_0 = 0$ lead to

$$\Phi_\varepsilon(\xi_t) \leq \Phi_\varepsilon(\xi_0)e^{Ct} = e^{Ct}, \quad t < \tau.$$

Letting $\varepsilon \downarrow 0$, we get $\Phi_0(\xi_t) \leq e^{Ct}$. If $\xi_t > 0$ for some given t, we have $\int_0^{\xi_t}\frac{ds}{s\log\frac{1}{s}} \leq Ct$, which is impossible. Therefore we must have $\xi_t = 0$, which means $X_t = Y_t$ for any $t \geq 0$. □

Choose $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi \geq 0, \quad \mathrm{supp}(\chi) \subset B(1), \quad \int_{\mathbb{R}^d}\chi dx = 1,$$

where $B(r) = \{x \in \mathbb{R}^d : |x| \leq r\}$. For $n \geq 1$, define $\chi_n(x) = 2^{dn}\chi(2^n x)$. Then $\mathrm{supp}(\chi_n) \subset B(2^{-n})$ and $\int_{\mathbb{R}^d}\chi_n dx = 1$. Set $V_n = V * \chi_n$ (convolution product), then V_n is a bounded smooth vector field on $\mathbb{R}^d$.

Proposition 1.3 *There exists $\zeta > 1$ such that*

$$\sup_{x\in\mathbb{R}^d}|V_n(x) - V(x)| \leq \zeta^{-n} \quad \textit{for } n \textit{ big enough.} \tag{1.2.3}$$

Proof. We have

$$\begin{aligned}|V_n(x) - V(x)| &\leq \int_{\mathbb{R}^d}|V(x-y) - V(y)|\chi_n(y)dy \leq C\int_{B(2^{-n})}|y|\log\frac{1}{|y|}\cdot\chi_n(y)dy \\ &\leq C2^{-n}\log 2^n \cdot \int_{B(2^{-n})}\chi_n(y)dy \leq C\zeta^{-n}\end{aligned}$$

for some $\zeta > 1$. □

Theorem 1.4 *Let $X_n(t,x)_{t\geq 0}$ be the solution to*

$$\frac{dX_n}{dt} = V_n(X_n), \quad X_n(0) = x.$$

Then for any $T > 0$,

$$\lim_{n\to\infty} \sup_{t\leq T} \sup_{x\in\mathbb{R}^d} |X_n(t,x) - X(t,x)| = 0. \tag{1.2.4}$$

Proof. For simplicity, we omit x in X_n as well as in X. Set $\xi_n(t) = |X_n(t) - X(t)|^2$ and

$$\tau_n = \inf\{t > 0 : \xi_n(t) \geq \delta^2\}.$$

Then by (1.2.1) and (1.2.3), for $t \leq \tau_n$,

$$\begin{aligned}\left|\frac{dX_n(t)}{dt} - \frac{dX(t)}{dt}\right| &\leq |V_n(X_n(t)) - V(X_n(t))| + |V(X_n(t)) - V(X(t))| \\ &\leq \zeta^{-n} + C|X_n(t) - X(t)| \log \frac{1}{|X_n(t) - X(t)|}.\end{aligned}$$

Therefore for $t \leq \tau_n$,

$$\left|\frac{d\xi_n(t)}{dt}\right| = 2\left|\langle X_n(t) - X(t), \frac{dX_n(t)}{dt} - \frac{dX(t)}{dt}\rangle\right| \leq 2\delta\zeta^{-n} + C\xi_n(t)\log\frac{1}{\xi_n(t)}.$$

Using the lemma below, we get for $t \leq \tau_n \wedge T$,

$$\xi_n(t) \leq (2\delta\zeta^{-n})^{e^{-Ct}} \leq (2\delta\zeta^{-n})^{e^{-CT}}.$$

This last quantity is less than δ^2 for n big enough. It follows that $\tau_n \geq T$ and

$$\sup_{0\leq t\leq T} \xi_n(t) \leq (2\delta\zeta^{-n})^{e^{-CT}}.$$

Letting $n \to \infty$ yields the result (1.2.4). □

A consequence of (1.2.4) is that $x \to X_t(x)$ defines a flow of homeomorphisms of $\mathbb{R}^d$. In fact, by previous section, the inverse maps X_n^{-1} as well as X_t^{-1} satisfy the same type differential equations. In the same way, X_n^{-1} converges to X_t^{-1} uniformly with respect to (t,x) in any compact subset of $[0,+\infty[\times\mathbb{R}^d$.

Lemma 1.5 *Let $\varphi : \mathbb{R}_+ \to (0,1)$ be a derivable function such that for $C > 0$,*

$$\varphi'(t) \leq C\varphi(t)\log\frac{1}{\varphi(t)}, \tag{1.2.5}$$

then

$$\varphi(t) \leq (\varphi(0))^{e^{-Ct}} \quad \textit{for } t \geq 0.$$

Proof. $\log\varphi(t)$ being negative, we use (1.2.5),

$$\frac{\varphi'(t)}{\varphi(t)\log\varphi(t)} \geq -C.$$

Integrating this inequality between $(0,t)$, it leads to

$$\int_{\varphi(0)}^{\varphi(t)} \frac{ds}{s\log s} \geq -Ct, \quad \text{or} \quad \log\left(\frac{\log\varphi(t)}{\log\varphi(0)}\right) \geq -Ct.$$

Therefore $\log\varphi(t) \leq \log\varphi(0)\cdot e^{-Ct}$ or $\varphi(t) \leq (\varphi(0))^{e^{-Ct}}$. □

In order to apply Lemma 1.5 in Theorem 1.4, we set

$$\eta_n(t) = \frac{2\delta}{C}\zeta^{-n} + \xi_n(t)$$

and observe that for δ small enough,

$$-C\xi_n(t)\log\xi_n(t) + 2\delta\zeta^{-n} \leq -C\eta_n(t)\log\eta_n(t).$$

□

Assume now the divergence $\mathrm{div}(V) \in L^1_{loc}(\mathbb{R}^d)$ exists in the distribution sense:

$$\int_{\mathbb{R}^d} \mathrm{div}(V)\,\varphi\,dx = -\int_{\mathbb{R}^d} \langle\nabla\varphi, V\rangle dx, \quad \varphi \in C_c^\infty(\mathbb{R}^d). \tag{1.2.6}$$

We have

$$\begin{aligned}\mathrm{div}(V^n) &= \sum_{i=1}^d \frac{\partial}{\partial x_i}(V^n)^i = \sum_{i=1}^d \int_{\mathbb{R}^d} V^i(y)\frac{\partial}{\partial x_i}\chi_n(x-y)dy \\ &= -\int_{\mathbb{R}^d} V(y)\cdot\nabla_y(\chi_n(x-y))dy = \int_{\mathbb{R}^d} \mathrm{div}(V)(y)\chi_n(x-y)dy = \mathrm{div}(V)*\chi_n,\end{aligned}$$

It follows that $\mathrm{div}(V_n)$ converges to $\mathrm{div}(V)$ in $L^1_{loc}(\mathbb{R}^d)$, as $n \to +\infty$.

Theorem 1.6 *Assume* (1.2.1) *and* $\mathrm{div}(V)$ *exists. Let* $\theta \in C(\mathbb{R}^d)$*. Then* $u_t(x) = \theta(X_t^{-1}(x))$ *satisfies the transport equation*

$$\frac{du_t}{dt} + V\cdot\nabla u_t = 0, \quad u_0 = \theta$$

in the sense that, for any $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$(u_t,\varphi)_{L^2} = (\theta,\varphi)_{L^2} - \int_0^t (u_s, \mathrm{div}(\varphi V))_{L^2} ds. \tag{1.2.7}$$

Proof. Step 1. Suppose firstly $\theta \in C^1$. Let X_n be given in Theorem 1.4 and set $u_n(t) = \theta(X_n^{-1}(t))$. Then by Section 1.1, for any $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$(u_n(t),\varphi)_{L^2} = (\theta,\varphi)_{L^2} - \int_0^t (u_n(s), \mathrm{div}(\varphi V_n))_{L^2} ds. \tag{1.2.8}$$

Let K be the support of φ, then the support of $\text{div}(\varphi V_n) = \varphi \text{div}(V_n) + \langle \nabla\varphi, V_n\rangle$ is contained in K. Let $R = \sup_{0\le t\le T} \sup_{x\in K} |X^{-1}(t,x)|$ which is finite. By (1.2.4), for n big enough, $X_n^{-1}(t,x) \in B(R+1)$ for all $0 \le t \le T$, $x \in K$. Therefore for $\varepsilon > 0$, there exists n_0 such that for $n \ge n_0$,

$$\sup_{x\in K} \sup_{t\le T} |\theta(X_n^{-1}(t,x)) - \theta(X^{-1}(t,x))| < \varepsilon.$$

Letting $n \to \infty$ in (1.2.8), we get (1.2.7).

Step 2. Let $\theta \in C(\mathbb{R}^d)$. Let $\theta_n \in C^1(\mathbb{R}^d)$ which converges to θ on any compact set. By Step 1, $u_t^n = \theta_n(X_t^{-1})$ satisfies (1.2.7). Now let $K = \text{supp}(\varphi)$. By what was done in the above, we see that $\theta_n(X_t^{-1})$ converges uniformly to $\theta(X_t^{-1})$ over K. So that letting $n \to \infty$ in

$$(u_t^n, \varphi)_{L^2} = (\theta_n, \varphi)_{L^2} - \int_0^t (u_s^n, \text{div}(\varphi V))_{L^2} ds,$$

we get the result. □

Corollary 1.7 *If $div(V) = 0$, then X_t preserves the Lebesgue measure.*

Proof. Take $\theta \in C_c(\mathbb{R}^d)$. Let $K = \text{supp}(\theta)$. Then $K_T = \cup_{0\le t\le T} X_t(K)$ is compact. Now for $x \in (K_T)^c$, then for any $t \in [0,T]$, $X_t^{-1}(x) \in K^c$, so that $\theta(X_t^{-1}(x)) = 0$. Now take $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi = 1$ on K_T. We have for $s \le t \le T$,

$$(u_s, \text{div}(\varphi V))_{L^2} = -\int u_s(x)\langle \nabla\varphi, V(x)\rangle dx = 0,$$

so that

$$\int_{\mathbb{R}^d} \theta(X_t^{-1}) dx = (u_t, \varphi)_{L^2} = (\theta, \varphi)_{L^2} = \int_{\mathbb{R}^d} \theta(x) dx,$$

which means that X_t^{-1} leaves the Lebesgue measure invariant, so does X_t □

For the general case, we have (see [10])

Theorem 1.8 *Assume* $\text{div}(V) \in L^\infty$. *Then the Lebesgue measure λ_d on $\mathbb{R}^d$ is quasi-invariant under the flow X_t: $(X_t)_*\lambda_d = k_t\,\lambda_d$; moreover*

$$e^{-t||\text{div}(V)||_\infty} \le k_t(x) \le e^{t||\text{div}(V)||_\infty}. \tag{1.2.9}$$

Proof. Take a positive function $\theta \in C_c(\mathbb{R}^d)$ and set $u_t = \theta(X_t^{-1})$. As seen in the proof of the above corollary, there exists $R > 0$ such that $u_t(x) = 0$ for $t \in [0,T]$ and $|x| > R$. Then for $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi(x) = 1$ for $|x| \le R$, we have, for $s \in [0,T]$, $(u_s, \text{div}(\varphi V))_{L^2} = -\int_{\mathbb{R}^d} u_s \,\text{div}(V)\, dx$. The equation (1.2.7) yields

$$\int_{\mathbb{R}^d} u_t\, dx = \int_{\mathbb{R}^d} \theta\, dx - \int_0^t \Big(\int_{\mathbb{R}^d} u_s \,\text{div}(V)\, dx\Big)\, ds.$$

It is easy to see that $t \to \int_{\mathbb{R}^d} u_t\, dx$ is absolutely continuous and

$$\Big|\frac{d}{dt}\int_{\mathbb{R}^d} u_t\, dx\Big| \le ||\text{div}(V)||_\infty \int_{\mathbb{R}^d} u_t\, dx.$$

We deduce that

$$e^{-t||\text{div}(V)||_\infty} \int_{\mathbb{R}^d} \theta\, dx \le \int_{\mathbb{R}^d} \theta(X_t^{-1})\, dx \le e^{t||\text{div}(V)||_\infty} \int_{\mathbb{R}^d} \theta\, dx.$$

It follows that $(X_t^{-1})_*\lambda_d$ is absolutely continuous with respect to λ_d. Now (1.2.9) follows. □

2 Diperna-Lions Theory

Let $1 \leq p \leq +\infty$. We denote by $L^p(\mathbb{R}^d)$ the space of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} |f(x)|^p \, dx < +\infty$, $L^p_{loc}(\mathbb{R}^d)$ the space of those functions f such that $f \mathbf{1}_K \in L^p(\mathbb{R}^d)$ for all compact subsets $K \subset \mathbb{R}^d$. Note that $L^p(\mathbb{R}^d)$ is a Banach space, having $L^q(\mathbb{R}^d)$ as the dual space, where $q \in [1, +\infty]$ is the conjugate number of p: $1/p + 1/q = 1$ and $L^p_{loc}(\mathbb{R}^d)$ is not a Banach space, but a vector space endowed with a complete metric. For a Banach space E, similarly, $L^p(\mathbb{R}^d, E)$ is the space of the measurable E-valued functions such that $\int_{\mathbb{R}^d} |f(x)|^p_E \, dx < +\infty$. The space $L^1([0,T], L^q_{loc})$ denotes the space of functions $(t,x) \to u_t(x)$ such that $\int_0^T \Big(\int_K |u_t(x)|^q dx \Big)^{1/q} dt$ is finite for any compact subset $K \subset \mathbb{R}^d$. For a sequence $u^n \in L^1([0,T], L^q_{loc})$, we say that it converges to $u \in L^1([0,T], L^q_{loc})$ if

$$\lim_{n \to +\infty} \int_0^T \Big(\int_K |u^n_t(x) - u_t(x)|^q dx \Big)^{1/q} dt = 0, \quad \text{for all compact } K \subset \mathbb{R}^d.$$

Definition 2.1 *Let $V(t,\cdot) \in L^1_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ be a time-dependent vector field and $c_t \in L^1_{loc}(\mathbb{R}^d)$. Let $p \in [1, \infty]$ and $T > 0$ be given. We say that $u \in L^\infty([0,T], L^p(\mathbb{R}^d))$ is a (weak) solution to*

$$\frac{\partial u_t}{\partial t} + V_t \cdot \nabla u_t + c_t u_t = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \tag{2.1}$$

with the initial function $u_0 \in L^p(\mathbb{R}^d)$, if for any $\phi \in C_0^\infty([0,T) \times \mathbb{R}^d)$,

$$-\int_0^T dt \int_{\mathbb{R}^d} u \frac{\partial \phi}{\partial t} dx - \int_{\mathbb{R}^d} u_0 \phi(0,x) dx + \int_0^T dt \int_{\mathbb{R}^d} u \, (-\mathrm{div}(V_t \phi) + c_t \phi) dx = 0. \tag{2.2}$$

Notice that the dependence of $t \to u_t$ is not necessarily continuous.

Remark 2.2 If we take $\phi(t,x) = \alpha(t)\phi(x)$ with $\alpha(0) = 0$ in (2.2), then

$$-\int_0^T \alpha'(t) dt \int_{\mathbb{R}^d} u\phi \, dx + \int_0^T \alpha(t) \int_{\mathbb{R}^d} u \, (-\mathrm{div}(V\phi) + c\,\phi) dx = 0,$$

or in the distribution sense,

$$\frac{d}{dt} \int_{\mathbb{R}^d} u_t \phi \, dx + \int_{\mathbb{R}^d} u_t \cdot (-\mathrm{div}(V\phi) + c\,\phi) dx = 0. \tag{2.3}$$

Note that $\mathrm{div}(V\phi) = \langle V, \nabla \phi \rangle + \phi \, \mathrm{div}(V)$. Therefore (2.2) makes sense provided we assume

$$c - \mathrm{div}(V) \in L^1([0,T], L^q_{loc}(\mathbb{R}^d)), \quad V \in L^1([0,T], L^q_{loc}(\mathbb{R}^d)). \tag{2.4}$$

Under the condition (2.4), there exists a constant $C_\phi > 0$ and $R > 0$ such that

$$\Big| \frac{d}{dt} \int_{\mathbb{R}^d} u_t \phi \, dx \Big| \leq C_\phi \Big[||c_t - \mathrm{div}(V_t)||_{L^q(B(R))} + ||v_t||_{L^q(B(R))} \Big].$$

We shall denote by $\xi(t) = \int_{\mathbb{R}^d} u_t \phi \, dx$, $m(t) = ||c_t - \mathrm{div}(V_t)||_{L^q(B(R))} + ||v_t||_{L^q(B(R))}$ and $\dot{\xi}$ the distributional derivative of ξ. Then we have

$$\int_0^T |\dot{\xi}(s)| \, ds \leq C_\phi \int_0^T m(s) \, ds < +\infty.$$

This means that ξ is in the Sobolev space $W^{1,1}([0,T])$. Now for a function $\gamma \in L^1([0,T])$, we say that $t_0 \in (0,T)$ is a Lebesgue point of γ if $\gamma(t_0) = \lim_{\eta\to 0} \frac{1}{2\eta}\int_{t_0-\eta}^{t_0+\eta} \gamma(s)\,ds$. By Lebesgue derivative theorem, the set of such points is of full measure. Now set

$$L = \{t \in (0,T);\ \text{Lebesgue point of } \xi\}.$$

Then for $t_1 < t_2$ in L, we have

$$\xi(t_2) - \xi(t_1) = \int_{t_1}^{t_2} \dot{\xi}(s)\,ds.$$

It follows that

$$|\xi(t_2) - \xi(t_1)| \le C_\phi \int_{t_1}^{t_2} m(s)\,ds, \quad t_1, t_2 \in L.$$

Since L is dense in $[0,T]$, we deduce that ξ admits an uniformly continuous extension $\tilde{\xi}$ on $[0,T]$. Again by above inequality, such an uniform continuous version is absolutely continuous; therefore $t \to \tilde{\xi}(t)$ is derivable at almost point of $(0,T)$.

Proposition 2.3 *Let $p \in [1,\infty]$ and $u_0 \in L^p(\mathbb{R}^d)$. Assume* (2.4) *and*

$$c,\, \mathrm{div}(V) \in L^1([0,T], L^\infty(\mathbb{R}^d)).$$

Then there exists a solution $u \in L^\infty([0,T], L^p(\mathbb{R}^d))$ to (2.1) *with u_0 given.*

Proof. For simplicity, consider the case $c = 0$. For $1 < p < \infty$, the function $x \to |x|^p$ is differentiable.
A priori estimate: if (2.1) is satisfied in the classical sense, using $(|x|^p)' = p\,\mathrm{sgn}(x)|x|^{p-1}$, then

$$\frac{\partial}{\partial t}|u_t|^p + V_t \cdot \nabla |u_t|^p = 0.$$

Integrating this equation over $\mathbb{R}^d$, we deduce

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u|^p dx = \int_{\mathbb{R}^d} \mathrm{div}(V_t)|u_t|^p dx \le \|\mathrm{div}(V_t)\|_\infty \int_{\mathbb{R}^d} |u_t|^p dx,$$

which implies by Gronwall lemma,

$$\int_{\mathbb{R}^d} |u_t|^p dx \le \left(\int_{\mathbb{R}^d} |u_0|^p dx\right) e^{\int_0^t \|\mathrm{div}(V_s)\|_\infty ds}. \tag{2.5}$$

The inequality (2.5) is the so-called a priori estimate on solutions to (2.1). Now we are going to prove the existence of solutions to (2.1) in the space $L^\infty([0,T], L^p)$. For simplicity, we suppose in the sequel that $V \in L^1([0,T], L^q)$ and consider the case $1 < p < +\infty$ (for other cases, we refer to [2]).

Let χ_n be a regularizing sequence: $\mathrm{supp}(\chi_n) \subset B(2^{-n})$, $\int_{\mathbb{R}^d} \chi_n(x)dx = 1$ and $\chi_n \ge 0$. Define

$$V_t^n = V_t * \chi_n, \quad u_0^n = u_0 * \chi_n.$$

We have

$$\|\chi_n\|_{L^1} = \|\chi\|_{L^1} = 1,\ \|u_0^n\|_{L^p} \le \|u_0\|_{L^p}\|\chi_n\|_{L^1} = \|u_0\|_{L^p}.$$

By harmonic analysis, as $n \to \infty$, for t fixed, $V_t^n \to V_t$ in L^q and $u_0^n \to u_0$ in L^p. Moreover, $V_t^n \in C_b^1(\mathbb{R}^d)$, $u_0^n \in C_b^1(\mathbb{R}^d)$. By Section 1, there exists a unique solution $u^n_\cdot \in C([0,T], C_b^1(\mathbb{R}^d))$ to

$$\frac{\partial u_t^n}{\partial t} + V_t^n \cdot \nabla u_t^n = 0, \quad u^n|_{t=0} = u_0^n.$$

Note that $\mathrm{div}(V_t^n) = \mathrm{div}(V_t) * \chi_n$, we have $\|\mathrm{div}(V_t^n)\|_\infty \le \|\mathrm{div}(V_t)\|_\infty$. Now applying (2.5) to u^n, we see that $(u^n)_{n\ge 1}$ is bounded in $L^\infty([0,T], L^p)$. Notice that $L^\infty([0,T], L^p)$ is the dual space of $L^1([0,T], L^q)$. Applying the following result in Functional Analysis, up to a subsequence, u^n converge to a function $u \in L^\infty([0,T], L^p)$ in the sense that

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} u^n(t)\varphi(t)dtdx = \int_0^T \int_{\mathbb{R}^d} u(t)\varphi(t)dtdx$$

for each $\varphi \in L^1([0,T], L^q)$. Now taking a limit procedure in (2.2), we see that $u \in L^\infty([0,T], L^p)$ is a solution to (2.2). □

Theorem*(from Functional Analysis). Let E be a Banach space; then the unit ball in the dual space E' is compact for weak* topology. More precisely, for any sequence $\ell_n \in E'$ such that $||\ell_n||_{E'} \le R$, then there exist $\ell \in E'$ of $||\ell||_{E'} \le R$ and a subsequence n_k such that*

$$\lim_{k\to+\infty} \ell_{n_k}(x) = \ell(x) \quad \textit{holds for all } x \in E.$$

It is often difficult to work with an equation in the distribution sense. The following approximation will play an important role.

Theorem 2.4 *Let $1 \le p \le \infty$ and q its conjugate number. Suppose furthermore that*

$$V \in L^1([0,T], W_{loc}^{1,\alpha}(\mathbb{R}^d)) \quad \textit{for some } \alpha \ge q. \tag{2.6}$$

Then, if we denote by $u_n = u * \chi_n$, u_n satisfies

$$\frac{\partial u_n}{\partial t} + V_t \cdot \nabla u_n = r_n, \tag{2.7}$$

where $r_n \to 0$ in $L^1([0,T], L_{loc}^\beta(\mathbb{R}^d))$ and β is given by

$$\begin{cases} \frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{p}, & \textit{if } \alpha \textit{ or } p < +\infty; \\ \textit{any } \beta < +\infty, & \textit{if } \alpha = \beta = \infty. \end{cases}$$

Proof. Recall that $f \in W_{loc}^{1,\alpha}(\mathbb{R}^d)$ if f and its distributional derivative $\partial f/\partial x_i$ are in $L_{loc}^\alpha(\mathbb{R}^d)$. Let $w \in L_{loc}^p(\mathbb{R}^d)$ and B be a vector field in $W_{loc}^{1,\alpha}$. Define the term $(B \cdot \nabla w) * \chi_n$ by

$$\begin{aligned} [(B\cdot\nabla w) * \chi_n](x) &= \int_{\mathbb{R}^d} (B\cdot\nabla w)\cdot \tau_x\chi_n\, dy = -\int_{\mathbb{R}^d} w\, \mathrm{div}(\tau_x\chi_n \cdot B)\, dy \\ &= -\int_{\mathbb{R}^d} w(y)\big(\chi_n(x-y)\mathrm{div}(B)(y) + \langle \nabla_y \chi_n(x-y), B(y)\rangle\big)\, dy, \end{aligned}$$

where $\tau_x f(y) = f(x-y)$.

Lemma 2.5 *For $B \in W^{1,\alpha}_{loc}(\mathbb{R}^d)$, set $r_n := (B \cdot \nabla w) * \chi_n - B \cdot \nabla(w * \chi_n)$. Then $r_n \to 0$ in $L^\beta_{loc}(\mathbb{R}^d)$.*

Proof. We have

$$r_n = -(w\,\mathrm{div}(B)) * \chi_n + \int_{\mathbb{R}^d} w(y)\langle B(y) - B(x), (\nabla\chi_n)(x-y)\rangle dy.$$

By hypothesis, $w\,\mathrm{div}(B) \in L^\beta_{loc}$; therefore $(w\,\mathrm{div}(B)) * \chi_n \to w\,\mathrm{div}(B)$ in L^β_{loc}.

Next we estimate the second term in the right hand of the above expression. Firstly we shall deal with the good case: w and $B \in C^1_b$. For n big enough, the integrand is concentrated in the ball $B(x, 2^{-n})$, therefore

$$\begin{aligned} I_n(x) &:= \int_{\mathbb{R}^d} w(y)\langle B(y) - B(x), (\nabla\chi_n)(x-y)\rangle dy \\ &\approx w(x)\int_{\mathbb{R}^d} \langle B(y) - B(x), -\nabla(\tau_x\chi_n)\rangle dy \\ &= w(x)\int_{\mathbb{R}^d} \mathrm{div}(B)\cdot\tau_x\chi_n dy = w(x)\int_{\mathbb{R}^d} \mathrm{div}(B)\cdot\chi_n(x-y)dy \\ &\to w(x)\mathrm{div}(B)(x) \quad \text{as } n \to \infty. \end{aligned}$$

Secondly, fix $R > 0$ and set $Q_x(y) = \langle B(y) - B(x), (\nabla\chi_n)(x-y)\rangle$. Then for $x \in B(R)$,

$$|I_n(x)|^\beta = \left|\int_{B(x,1)} w(y)Q_x(y)dy\right|^\beta \le [\lambda(B(x,1))]^{\beta-1}\int_{B(R+1)} |w(y)Q_x(y)|^\beta dy,$$

where λ denotes the Lebesgue measure. Let $C_\beta = \lambda(B(0,1))^{(\beta-1)/\beta}$. Then, according to Hölder inequality,

$$|I_n(x)| \le C_\beta \|w\|_{L^p(B(R+1))}\left(\int |Q_x(y)|^\alpha dy\right)^{\frac{1}{\alpha}}.$$

We have

$$|Q_x(y)| \le \frac{|B(y) - B(x)|}{2^{-n}}|\nabla\chi(\frac{x-y}{2^{-n}})| \le ||\nabla\chi||_\infty \frac{|B(y) - B(x)|}{2^{-n}}|\,\mathbf{1}_{\{|y-x|\le 2^{-n}\}}.$$

It follows that there exists a constant $C_{\beta,\alpha,R} > 0$ such that

$$\|I_n\|_{L^\beta(B(R))} \le C_{\beta,\alpha,R}\,\|w\|_{L^p(B(R+1))}\left[\int_{B(R)}\left(\int_{|x-y|\le 2^{-n}}\left(\frac{|B(y) - B(x)|}{2^{-n}}\right)^\alpha dy\right)dx\right]^{\frac{1}{\alpha}}.$$

Now we claim that

$$\left[\int_{B(R)}\left(\int_{|x-y|\le 2^{-n}}\left(\frac{|B(y) - B(x)|}{2^{-n}}\right)^\alpha dy\right)dx\right]^{\frac{1}{\alpha}} \le \|\nabla B\|_{L^\alpha(B(R+1))}. \tag{2.8}$$

If $B \in C^1$, then

$$|B(y) - B(x)| \le |y - x|\int_0^1 |\nabla B(x + t(y-x))|dt$$

and

$$\int_{|x-y|\leq 2^{-n}} \left(\frac{|B(y)-B(x)|}{2^{-n}}\right)^{\alpha} dy \leq \int_{|x-y|\leq 2^{-n}} \int_0^1 |\nabla B(x+t(y-x))|^{\alpha} dt dy.$$

Therefore

$$\begin{aligned}
&\int_{B(R)} \left(\int_{|x-y|\leq 2^{-n}} \left(\frac{|B(y)-B(x)|}{2^{-n}}\right)^{\alpha} dy\right) dx \\
&\leq \int_{B(R)} dx \int_{|x-y|\leq 2^{-n}} \int_0^1 |\nabla B(x+t(y-x))|^{\alpha} dt dy \\
&= \int_{B(R)} dx \int_{B(2^{-n})} \int_0^1 |\nabla B(x+tz)|^{\alpha} dt dz \\
&\leq \int_{B(R)} dx \int_{B(1)} \int_0^1 |\nabla B(x)|^{\alpha} dt dz = \lambda(B(1)) \|\nabla B\|^{\alpha}_{L^{\alpha}(B(R+1))}.
\end{aligned}$$

We get (2.8). Now by density argument, we see that (2.8) holds for $w \in L^p_{loc}(\mathbb{R}^d)$ and $B \in W^{1,\alpha}_{loc}$. Finally

$$||I_n||_{L^{\beta}(B(R))} \leq C_{\beta,\alpha,R} \, ||w||_{L^p(B(R+1))} \, ||\nabla B||_{L^{\alpha}(B(R+1))}.$$

Using this estimate and the first step, we can see that $I_n \to w \,\mathrm{div}(B)$ in L^{β}_{loc} for $B \in W^{1,\alpha}_{loc}$. □

End of the proof of Theorem 2.4. Since $u_n = u * \chi_n$, $\frac{\partial u_n}{\partial t} = \frac{\partial u}{\partial t} * \chi_n$ and

$$V_t \cdot \nabla u_n = V_t \cdot \nabla(u * \chi_n) = (V_t \cdot \nabla u) * \chi_n + r_n(t).$$

We have

$$\frac{\partial u_n}{\partial t} + V_t \cdot \nabla u_n = \left(\frac{\partial u}{\partial t} + V_t \cdot \nabla u\right) * \chi_n + r_n(t) = r_n(t).$$

Now using the above observation, we get the result. □

Remark 2.6 The equation (2.7) is understood in the distribution sense. However by the equation, we see that $u_n \in W^{1,1}_{loc}((0,T) \times \mathbb{R}^d)$; therefore for a.e. $x \in \mathbb{R}^d$, $t \to u_n(t,x)$ is derivable at a.e. $t_0 \in (0,T)$ in the classical sense.

In the sequel, we shall say that $u \in L^1([0,T], L^1) + L^1([0,T], L^{\infty})$ if $u = u_1 + u_2$ such that u_1 belongs to the first space and u_2 to the second one.

Theorem 2.7 *(Uniqueness) Under the hypothesis that*

$$V \in L^1([0,T], W^{1,q}_{loc}), \quad \mathrm{div}(V) \in L^1([0,T], L^{\infty}(\mathbb{R}^d))$$

and

$$\frac{V}{1+|x|} \in L^1([0,T], L^1) + L^1([0,T], L^{\infty}),$$

the uniqueness holds in $L^{\infty}([0,T], L^p)$, *where* $p \in [1, \infty]$.

Proof. We prove only the case $p > 1$. For $p = 1$, we refer to [2]. Let $u \in L^\infty([0,T], L^p)$ be a solution to (2.1) such that $u|_{t=0} = 0$. Let $u_n = u * \chi_n$. Using Theorem 2.4,

$$r_n = \frac{\partial u_n}{\partial t} + V \cdot \nabla u_n \to 0 \quad \text{in } L^1([0,T], L^1_{loc}(\mathbb{R}^d)).$$

Let $\beta \in C^1(\mathbb{R})$ with bounded derivative. We have

$$\frac{\partial}{\partial t}\beta(u_n) + V \cdot \nabla\, \beta(u_n) = r_n \beta'(u_n). \tag{2.9}$$

As $|\beta(u_n) - \beta(u)| \leq \|\beta'\|_\infty |u_n - u|$, $\beta(u_n)$ converges to $\beta(u)$ in L^p. Letting $n \to \infty$ in (2.9) yields

$$\frac{\partial}{\partial t}\beta(u) + V \cdot \nabla\beta(u) = 0 \quad \text{in distribution sense.} \tag{2.10}$$

Let $\phi \in C_c^\infty(\mathbb{R}^d)$. By the remark at the beginning of this section, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(u)\,\phi\, dx = \int_{\mathbb{R}^d} \mathrm{div}(V_t\phi)\beta(u)\, dx.$$

Take now $\Phi \in C_0^\infty(\mathbb{R}^d)$, $0 \leq \Phi \leq 1$, $\mathrm{supp}(\Phi) \subset B(2)$ and $\Phi \equiv 1$ on $B(1)$. Next, we consider $\Phi_R(x) = \Phi\big(x/R\big)$, $R \geq 1$. Then

$$\frac{d}{dt}\int \beta(u)\Phi_R dx - \int \beta(u)\mathrm{div}(V)\Phi_R dx = \int \beta(u) V \cdot \nabla\Phi_R dx.$$

Let $M > 0$, put $\beta(t) = (|t| \wedge M)^p$ (β is Lipschitz continuous but not in C^1: this problem may be overcome by approximation argument), then

$$\frac{d}{dt}\int (|u| \wedge M)^p \Phi_R dx \leq C_t \int (|u| \wedge M)^p \Phi_R dx + \frac{\|\nabla\Phi\|_\infty}{R}\int_{R \leq |x| \leq 2R} (|u| \wedge M)^p |V(t,x)| dx.$$

Next we observe that $(|u| \wedge M)^p \in L^\infty([0,T], L^1 \cap L^\infty)$ while

$$\frac{1}{R}|V(t,x)|\mathbf{1}_{\{R \leq |x| \leq 2R\}} \leq \frac{4|V(t,x)|}{1+|x|}\mathbf{1}_{\{R \leq |x|\}},$$

therefore

$$\frac{d}{dt}\int (|u| \wedge M)^p \Phi_R dx \leq C_t \int (|u| \wedge M)^p \Phi_R dx \\ + m(t)\int_{|x| \geq R} (|u| \wedge M)^p\, dx + CM^p \int_{|x| \geq R} \frac{|V_1(t,x)|}{1+|x|}\, dx,$$

where $m(t) = \left\| \frac{V_2(t,x)}{1+|x|} \right\|_\infty$. Let $D_1(R) = \int_{|x| \geq R} (|u| \wedge M)^p\, dx$ and $D_2(R) = CM^p \int_{|x| \geq R} \frac{|V_1(t,x)|}{1+|x|}\, dx$. Then

$$\frac{d}{dt}\int (|u| \wedge M)^p \Phi_R dx \leq C_t \int (|u| \wedge M)^p \Phi_R dx + m(t)D_1(R) + D_2(R).$$

Gronwall lemma yields

$$\int (|u| \wedge M)^p \Phi_R dx \leq \int_0^t e^{\int_s^t c_u du} \left(D_1(R) m(s) + D_2(R) \right) ds.$$

Notice that $\frac{V_1}{1+|x|} \in L^1([0,T], L^1)$ and $\lim_{R\to+\infty} D_1(R) = \lim_{R\to+\infty} D_2(R) = 0$. Letting $R \to \infty$ in the above inequality yields $\int (|u| \wedge M)^p \, dx = 0$ which implies that $|u| \wedge M = 0$ almost everywhere. Letting $M \uparrow +\infty$ gives the result. □

In what follows, we shall indicate how to reduce c in (2.1) to the case $c = 0$. First, notice that if u_t solves

$$\frac{\partial u_t}{\partial t} + V_t \cdot \nabla u_t = c_t, \tag{2.11}$$

then

$$\begin{aligned} \frac{d}{dt}\big[u_t(X_t)\big] &= (\tfrac{d}{dt} u_t)(X_t) + (\nabla u_t)(X_t) \cdot \tfrac{dX_t}{dt} \\ &= \tfrac{d}{dt} u_t + \nabla u_t \cdot V_t = c_t(V_t), \end{aligned}$$

where X_t is the flow associated to V_t. It follows that $u_t(X_t) = c_0 + \int_0^t c_s(X_s)\, ds$ or

$$u_t = c_0(X_t^{-1}) + \int_0^t c_s(X_{t-s}^{-1})\, ds. \tag{2.12}$$

So the flow X_t allows also to solve (2.11) via (2.12). Secondly, let w_t be a solution to

$$\frac{dw_t}{dt} + V_t \cdot \nabla w_t = 0. \tag{2.13}$$

Consider $\tilde{w}_t = e^{-u_t} w_t$. Then $\frac{\partial \tilde{w}_t}{\partial t} = e^{-u_t} \frac{dw_t}{dt} - e^{-u_t} w_t \frac{\partial u_t}{\partial t}$ and $\nabla \tilde{w}_t = e^{-u_t} \nabla w_t - e^{-u_t} w_t \nabla u_t$. Therefore, according to (2.11) and (2.13),

$$\frac{\partial \tilde{w}_t}{\partial t} + \nabla \tilde{w}_t \cdot V_t = -c_t \tilde{w}_t,$$

or $\tilde{w}_t$ solves (2.1).

To conclude this section, we state the following result

Theorem 2.8 *Under the hypothesis that*

$$V \in L^1([0,T], W^{1,q}_{loc}), \quad c, \mathrm{div}(V) \in L^1([0,T]), L^\infty(\mathbb{R}^d)$$

and

$$\frac{V}{1+|x|} \in L^1([0,T], L^1(\mathbb{R}^d)),$$

the transport equation (2.1) admits a unique solution $u \in L^\infty([0,T], L^p)$ *if* $u_0 \in L^p$*, where* $p \in [1, +\infty]$.

3 Ambrosio's Representation Formula

Denote by

$$\mathcal{M}_+(\mathbb{R}^d) = \{\text{positive regular Borel Radon measure on } \mathbb{R}^d\}.$$

A measure $\mu \in \mathcal{M}_+(\mathbb{R}^d)$ is locally finite, the suitable topology on it is vague convergence: $\mu_n \to \mu$ vaguely if for any $f \in C_c(\mathbb{R}^d)$, $\int f d\mu_n \to \int f d\mu$. Consider now

$$\mathcal{M}_+^f(\mathbb{R}^d) = \{\mu \in \mathcal{M}_+(\mathbb{R}^d) : \mu(\mathbb{R}^d) < +\infty\}.$$

There are two natural topologies on $\mathcal{M}_+^f(\mathbb{R}^d)$:

(i) Narrow convergence: $\int f d\mu_n \to \int f d\mu$ for any $f \in C_b(\mathbb{R}^d)$;

(ii) w^*-convergence: $\int f d\mu_n \to \int f d\mu$ for any $f \in C_0(\mathbb{R}^d)$.

It is known that μ_n converges narrowly to μ if and only if μ_n converges to μ for w^* topology and $\mu_n(\mathbb{R}^d) \to \mu(\mathbb{R}^d)$. In particular, if we denote by $\mathcal{P}(\mathbb{R}^d)$ the space of probability measures on $\mathbb{R}^d$, then the weak convergence in $\mathcal{P}(\mathbb{R}^d)$ is identical to the narrow convergence, as well as the convergence for w^* topology.

Note that the notion of narrow convergence can be defined on any Polish space E and the following result will be used freely.

Prohokov Theorem. *Let $(\mu_n)_{n\geq 1} \subset \mathcal{M}_+^f(E)$ such that $\sup_{n\geq 1} \mu_n(E) < +\infty$. Then the tightness implies the relative compactness with respect to the narrow convergence.*

Definition 3.1 *Let $(t, x) \to V_t(x) \in \mathbb{R}^d$ a time-dependent Borel vector fields. We say that a family of measures $(\mu_t)_{t\in[0,T]}$ on $\mathbb{R}^d$ is a solution to the continuity equation*

$$\frac{d\mu_t}{dt} + D_x \cdot (V_t \mu_t) = 0, \quad \textit{in } [0, T] \times \mathbb{R}^d, \tag{3.1}$$

if $\int_0^T \Big(\int_K |V_t|\, d\mu_t \Big) dt < +\infty$ for any compact subset K of $\mathbb{R}^d$ and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\frac{d}{dt} \int_{\mathbb{R}^d} \varphi\, d\mu_t = \int_{\mathbb{R}^d} V_t \cdot \nabla\varphi\, d\mu_t. \tag{3.2}$$

Note. $D_x\cdot$ is the formal divergence.

Theorem 3.2 *Let $(\mu_t)_{t\in(0,T)}$ be a solution to (3.1) such that $\sup_{t\in[0,T]} \mu_t(\mathbb{R}^d) < +\infty$; then it admits a version $(\tilde{\mu}_t)_{t\in[0,T]}$ such that $t \to \tilde{\mu}_t$ is continuous respect to the w^* topology.*

Proof. The left hand side in (3.2) is taken in distribution sense and

$$\left| \frac{d}{dt} \int_{\mathbb{R}^d} \varphi\, d\mu_t \right| \leq C_\varphi \int_K |V_t|\, d\mu_t \in L^1([0, T]).$$

So $t \to \int \varphi\, d\mu_t$ is in $W^{1,1}([0, T])$. By discussion in Remark 2.2, it admits a uniformly continuous version: there is a full measure subset L_φ of $(0, T)$ and a uniformly continuous function ξ_φ on $[0, T]$ such that $\int \varphi\, d\mu_t = \xi_\varphi(t)$ for $t \in L_\varphi$. Take a countable subset $D \subset C_c^\infty(\mathbb{R}^d)$ which is dense in $C_c(\mathbb{R}^d)$ and set $L = \cap_{\varphi\in D} L_\varphi$. Let $\phi \in C_c(\mathbb{R}^d)$ and $\varepsilon > 0$, there is $\varphi \in D$ such that $||\varphi - \phi||_\infty < \varepsilon$. For $s, t \in L$, $|\int \phi\, d\mu_t - \int \phi\, d\mu_s| \leq |\int \varphi\, d\mu_t - \int \varphi\, d\mu_s| + 2C\varepsilon$, where C is a constant dominating all $\mu_t(\mathbb{R}^d)$. Therefore $t \to \int \phi\, d\mu_t$ is uniformly continuous over L. Let ξ_ϕ be the uniform continuous extension over $[0,T]$. We have for $t \in L$

$$|\xi_\phi(t)| = \left| \int \phi\, d\mu_t \right| \leq C\, ||\phi||_\infty.$$

This inequality extends to $[0,T]$. By Riesez representation theorem, for each $t \in [0,T]$, there is a Borel measure $\tilde{\mu}_t$ such that $\xi_\phi(t) = \int \phi\, d\tilde{\mu}_t$. Therefore $(\mu_t)_{t\in[0,T]}$ admits a version of $(\tilde{\mu}_t)_{t\in[0,T]}$, which is continuous with respect to the w^* topology. □

Remark 3.3 Suppose that

$$\int_0^T \int_{\mathbb{R}^d} \frac{|V_t(x)|}{1+|x|} d\mu_t(x) dt < +\infty, \tag{3.3}$$

then $\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$ for all $t \in [0,T]$.

In fact, consider a function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi \equiv 1$ on $B(1)$ and $\varphi \equiv 0$ on $(B(2))^c$. Define $\varphi_R(x) = \varphi(x/R)$, we have

$$\|\nabla \varphi_R\|_\infty \leq \frac{\|\nabla\varphi\|_\infty}{R} \mathbf{1}_{\{R\leq |x| \leq 2R\}} \leq \frac{4\|\nabla\varphi\|_\infty}{1+|x|} \mathbf{1}_{\{|x|\geq R\}}.$$

Therefore

$$\left| \frac{d}{dt} \int \varphi_R\, d\mu_t \right| \leq 4\|\nabla\varphi\|_\infty \int_{|x|\geq R} \frac{|V_t(x)|}{1+|x|} d\mu_t(x) dt,$$

or $\int \varphi_R\, d\mu_t = \int \varphi_R\, d\mu_0 + \varepsilon_R$, where

$$\varepsilon_R \leq \int_0^T \int_{|x|\geq R} \frac{|V_t(x)|}{1+|x|} d\mu_t(x) dt \to 0 \quad \text{as } R \to +\infty.$$

Then letting $R \to +\infty$ gives the result. □

Remark 3.4 Let V_t be a smooth vector field such that

$$\frac{dX_t}{dt} = V_t(X_t)$$

is well defined over $[0,T]$. For μ_0 a probability measure on $\mathbb{R}^d$, set

$$\mu_t = (X_t)_* \mu_0.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then

$$\frac{d}{dt} \int \varphi\, d\mu_t = \frac{d}{dt} \int \varphi(X_t) d\mu_0 = \int \langle \nabla\varphi(X_t), V_t(X_t)\rangle d\mu_0 = \int \langle \nabla\varphi, V_t\rangle d\mu_t.$$

Therefore $(\mu_t)_{t\geq 0}$ satisfies the continuity equation

$$\frac{\partial \mu_t}{\partial t} + D_x \cdot (V_t \mu_t) = 0 \quad \text{with } \mu|_{t=0} = \mu_0.$$

□

Remark 3.5 If $V \in L^1([0,T], L^q_{loc})$ and $\mathrm{div}(V) \in L^1([0,T], L^\infty)$ and $\mu_t = \rho_t \lambda_d$ with $\rho_t \in L^\infty([0,T], L^p_{loc})$, then (3.1) is reduced to the transport equation

$$\frac{\partial \rho_t}{\partial t} + V_t \cdot \nabla \rho_t + \mathrm{div}(V_t)\rho_t = 0. \tag{3.4}$$

In the sequel, we shall always consider the family of *probability measures* $(\mu_t)_{t\in[0,T]}$. According to Theorem 3.2, such a solution to the continuity equation (3.1) admits a version $(\tilde{\mu}_t)_{t\in[0,T]}$, which is weakly continuous.

Now let $W = C([0,T], \mathbb{R}^d)$, $W_x = \{\gamma \in W : \gamma(0) = x\}$ and $e_t : W \to \mathbb{R}^d$ be the evaluation map: $e_t(\gamma) = \gamma(t)$. Given an $\eta \in \mathcal{P}(\mathbb{R}^d \times W)$, we define $\mu_t^\eta = (e_t)_* \eta$ for $t \in [0,T]$. Let $\eta_x(d\gamma)$ be the conditional probability measure given $e_0 = x$; then

$$\int_{\mathbb{R}^d \times W} \psi(x,\gamma) d\eta(x,\gamma) = \int_{\mathbb{R}^d} \left(\int_{W_x} \psi(x,\gamma) d\eta_x(\gamma) \right) d\mu_0^\eta(x).$$

Theorem 3.6 *Let $(\mu_t)_{t\in[0,T]}$ be a solution to (3.1). Assume that μ_0 admits the finite first moment and*

$$\int_0^T \int_{\mathbb{R}^d} \frac{|V_t(x)|^\alpha}{1+|x|} d\mu_t(x) dt < +\infty \quad \text{for some } \alpha > 1. \tag{3.5}$$

Then there exists $\eta \in \mathcal{P}(\mathbb{R}^d \times W)$ such that $\mu_t^\eta = \mu_t$ for $t \in [0,T]$ and under η it holds

$$\gamma(t) = x + \int_0^t V_s(\gamma(s)) ds. \tag{3.6}$$

Proof. The proof consists of several steps.

Step 1 (smoothing): Let $\rho_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}}$. Recall that $(\mu_0 * \rho_\varepsilon)(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(x-y)\, d\mu_0(y)$, which is a smooth function. It is easy to check that

$$\sup_{0<\varepsilon\le 1} \left(\int_{\mathbb{R}^d} |x| \cdot (\mu_0 * \rho_\varepsilon)(x) dx \right) \le 1. \tag{i}$$

Similarly $(V_t \mu_t) * \rho_\varepsilon(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(x-y) V_t(y) d\mu_t(y)$, which is a smooth vector field. Define

$$\mu_t^\varepsilon = \mu_t * \rho_\varepsilon, \quad V_t^\varepsilon = \frac{(V_t\mu_t) * \rho_\varepsilon}{\mu_t^\varepsilon}.$$

Then $\mu_t^\varepsilon \to \mu_t$ weakly as $\varepsilon \downarrow 0$. We have for $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\int \varphi\, D_x \cdot (V_t^\varepsilon \mu_t^\varepsilon)\, dx &= -\int \langle \nabla\varphi, V_t^\varepsilon \rangle \mu_t^\varepsilon\, dx \\ &= -\int \langle \nabla\varphi, (V_t\mu_t) * \rho_\varepsilon \rangle\, dx = \int \varphi\, D_x \cdot [(V_t\mu_t) * \rho_\varepsilon]\, dx.\end{aligned}$$

It follows that $D_x \cdot (V_t^\varepsilon \mu_t^\varepsilon) = D_x \cdot [(V_t\mu_t) * \rho_\varepsilon]$ and

$$\begin{aligned}\frac{d}{dt}\mu_t^\varepsilon + D_x \cdot (V_t^\varepsilon \mu_t^\varepsilon) &= \left(\frac{d}{dt}\mu_t\right) * \rho_\varepsilon + D_x \cdot (V_t\mu_t) * \rho_\varepsilon \\ &= \left[\frac{d}{dt}\mu_t + D_x \cdot (V_t\mu_t)\right] * \rho_\varepsilon = 0,\end{aligned}$$

or μ_t^ε solves the transport equation

$$\frac{\partial \mu_t^\varepsilon}{\partial t} + V_t^\varepsilon \cdot \nabla \mu_t^\varepsilon + \mathrm{div}(V_t^\varepsilon)\mu_t^\varepsilon = 0.$$

On the other hand, let $X^\varepsilon(t,\cdot)$ be the solution to

$$\frac{dX^\varepsilon(t,\cdot)}{dt} = V_t^\varepsilon(X^\varepsilon(t,\cdot)).$$

Consider: $\nu_t^\varepsilon = (X_t^\varepsilon)_*\mu_0^\varepsilon$. By Remark 3.4 and 3.5, the density of ν_t^ε solves the same above transport equation with the same initial data as μ_t^ε. By uniqueness, we get $\mu_t^\varepsilon = \nu_t^\varepsilon$ for all t. Define

$$\eta^\varepsilon = (I \times X^\varepsilon)_*\mu_0^\varepsilon, \quad \text{where } I \times X^\varepsilon : x \to (x, X^\varepsilon(\cdot, x)) \in \mathbb{R}^d \times W.$$

Then

$$\int_{\mathbb{R}^d \times W} \varphi(\gamma(t)) d\eta^\varepsilon = \int_{\mathbb{R}^d} \varphi(X^\varepsilon(t,x)) d\mu_0^\varepsilon = \int \varphi \, d\mu_t^\varepsilon. \tag{3.7}$$

Step 2 (tightness): Now we are going to prove that the family $\{\eta^\varepsilon : \varepsilon > 0\}$ is tight. Consider the functional ψ on $\mathbb{R}^d \times W$:

$$\psi(x,\gamma) = \begin{cases} |x| + \int_0^T \frac{|\dot\gamma(t)|^\alpha}{1+|\gamma(t)|} dt, & \text{if } \gamma \text{ is absolutely continuous;} \\ +\infty, & \text{otherwise.} \end{cases}$$

where $\alpha > 1$. We claim that $K_M = \{(x,\gamma) : \ \gamma(0) = x, \psi(x,\gamma) \le M\}$ is a compact subset of $\mathbb{R}^d \times W$. In fact, for $(x,\gamma) \in K_M$, we have $\gamma(t) = x + \int_0^t \dot\gamma(s) ds$, hence

$$\begin{aligned} |\gamma(t)| &\le |x| + \int_0^t |\dot\gamma(s)| ds \\ &\le |x| + \left(\int_0^t \frac{|\dot\gamma(s)|^\alpha}{1+|\gamma(s)|} ds \right)^{\frac{1}{\alpha}} \left(\int_0^t (1 + |\gamma(s)|)^{\frac{\beta}{\alpha}} ds \right)^{\frac{1}{\beta}} \\ &\le M + M^{\frac{1}{\alpha}} T^{\frac{1}{\beta}} (1 + \|\gamma\|_\infty)^{\frac{1}{\alpha}}, \end{aligned}$$

which implies that

$$\|\gamma\|_\infty \le M + M^{\frac{1}{\alpha}} T^{\frac{1}{\beta}} (1 + \|\gamma\|_\infty)^{\frac{1}{\alpha}}.$$

Therefore it exists a constant $C = C(M, \alpha, T)$ such that

$$1 + \|\gamma\|_\infty \le C(1 + \|\gamma\|_\infty)^{\frac{1}{\alpha}}$$

or $1 + \|\gamma\|_\infty \le C^{\frac{\alpha}{\alpha-1}}$. This means that $\{\gamma : (\gamma(0), \gamma) \in K_M\}$ is uniformly bounded. Now

$$|\gamma(t) - \gamma(s)| \le \left(\int_s^t \frac{|\dot\gamma(u)|^\alpha}{1+|\gamma(u)|} du \right)^{\frac{1}{\alpha}} \left(\int_s^t (1 + |\gamma(u)|)^{\frac{\beta}{\alpha}} du \right)^{\frac{1}{\beta}} \le C \, |t-s|^{\frac{1}{\beta}},$$

where $C > 0$ is a constant. By Ascoli theorem, we see that K_M is compact.
Now

$$\int_{\mathbb{R}^d} \psi(x,\gamma) d\eta^\varepsilon(x,\gamma) = \int |x| d\mu_0^\varepsilon + \int_0^T \int_{\mathbb{R}^d} \frac{|\dot X^\varepsilon(t,x)|^\alpha}{1 + |X^\varepsilon(t,x)|} d\mu_0^\varepsilon(x) dt.$$

The second term on the right hand side is equal to

$$\int_0^T\int_{\mathbb{R}^d}\frac{|v_t^\varepsilon(X^\varepsilon(t,x))|^\alpha}{1+|X^\varepsilon(t,x)|} = \int_0^T\int_{\mathbb{R}^d}\frac{|V_t^\varepsilon(x)|^\alpha}{1+|x|}d\mu_t^\varepsilon(x)dt.$$

Note that

$$|V_t^\varepsilon(x)|^\alpha \le \left(\frac{\int\rho_\varepsilon(x-y)|V_t(y)|d\mu_t(y)}{\int\rho_\varepsilon(x-y)d\mu_t(y)}\right)^\alpha \le \frac{\int\rho_\varepsilon(x-y)|V_t(y)|^\alpha d\mu_t(y)}{\mu_t^\varepsilon(x)},$$

so that the above quantity is dominated by

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|V_t(y)|^\alpha\rho_\varepsilon(x-y)}{1+|x|}d\mu_t(y)dxdt &= \int_0^T\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\frac{\rho_\varepsilon(x-y)}{1+|x|}dx\right)|V_t(y)|^\alpha d\mu_t(y)dt\\ &\to \int_0^T\int_{\mathbb{R}^d}\frac{|V_t(y)|^\alpha}{1+|y|}d\mu_t(y)dt < +\infty \quad \text{as } \varepsilon\downarrow 0.\end{aligned}$$

Therefore

$$\sup_{0<\varepsilon\le 1}\int\psi(x,\gamma)d\eta^\varepsilon(x,\gamma) < +\infty,$$

which implies that $\{\eta^\varepsilon : \varepsilon > 0\}$ is tight. Now let η be a limit point. By (3.7), we have

$$\int_{\mathbb{R}^d\times W}\varphi(\gamma(t))d\eta = \int\varphi\, d\mu_t. \tag{3.8}$$

Final Step. The condition (3.5) implies that $\int_0^T\int_{\mathbb{R}^d}\frac{|V_t(x)|}{1+|x|}\,d\mu_t(x)dt < +\infty$. Fix $\varepsilon_0\in(0,1/2)$; introduce the measure ν on $[0,T]\times\mathbb{R}^d$ by

$$\int\varphi(t,x)d\nu(t,x) = \int_0^T\int_{\mathbb{R}^d}\frac{\varphi(t,x)}{1+|x|-\varepsilon_0}\,d\mu_t(x)dt.$$

Note that ν is not a probability measure, but finite. It is clear that $V\in L^1([0,T]\times\mathbb{R}^d,\nu)$. Pick a sequence $\{C^n\}_{n\ge1}$ of continuous functions with compact support, converging to V in $L^1([0,T]\times\mathbb{R}^d,\nu)$. We have for such a C^n and $t\in[0,T]$,

$$\begin{aligned}&\int_{\mathbb{R}^d\times W}\frac{|\gamma(t)-x-\int_0^t C_s^n(\gamma(s))ds|}{1+\|\gamma\|_\infty}\,d\eta^\varepsilon\\ &=\int_{\mathbb{R}^d}\frac{|X^\varepsilon(t,x)-x-\int_0^t C_s^n(X^\varepsilon(s,x))ds|}{1+\|X^\varepsilon(\cdot,x)\|_\infty}\,d\mu_0^\varepsilon(x)\\ &=\int_{\mathbb{R}^d}\frac{|\int_0^t\big(V_s^\varepsilon(X^\varepsilon(s,x))-C_s^n(X^\varepsilon(s,x))\big)ds|}{1+\|X^\varepsilon(\cdot,x)\|_\infty}\,d\mu_0^\varepsilon(x)\\ &\le\int_0^t\int_{\mathbb{R}^d}\frac{|V_s^\varepsilon-C_s^n|}{1+|x|}\,d\mu_s^\varepsilon ds\\ &\le\int_0^t\int_{\mathbb{R}^d}\frac{|V_s^\varepsilon-C_s^{n,\varepsilon}|}{1+|x|}\,d\mu_s^\varepsilon ds+\int_0^t\int_{\mathbb{R}^d}\frac{|C_s^{n,\varepsilon}-C_s^n|}{1+|x|}\,d\mu_s^\varepsilon ds\end{aligned}$$

where $C_s^{n,\varepsilon}$ is defined in the same way as V_s^ε. Since $C_s^{n,\varepsilon}\to C_s^n$ uniformly as $\varepsilon\downarrow 0$,

$$\lim_{\varepsilon\to0}\int_0^t\int_{\mathbb{R}^d}\frac{|C_s^{n,\varepsilon}-C_s^n|}{1+|x|}\,d\mu_s^\varepsilon ds = 0.$$

Remark that

$$\int_0^t\int_{\mathbb{R}^d}\frac{|V_s^\varepsilon - C_s^{n,\varepsilon}|}{1+|x|}\,d\mu_s^\varepsilon ds \le \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|V_s(y)-C_s^n(y)|}{1+|x|}\rho_\varepsilon(x-y)\,d\mu_s(y)dxds,$$

and for $\varepsilon \le \varepsilon_0$,

$$\int_{\mathbb{R}^d}\frac{\rho_\varepsilon(x-y)}{1+|x|}\,dx = \int_{\mathbb{R}^d}\frac{\rho_\varepsilon(x)}{1+|x+y|}\,dx \le \frac{1}{1+|y|-\varepsilon_0}.$$

Finally

$$\int_{\mathbb{R}^d\times W}\frac{|\gamma(t)-x-\int_0^t C_s^n(\gamma(s))ds|}{1+\|\gamma\|_\infty}\,d\eta^\varepsilon \le ||V-C^n||_{L^1(\nu)} + r_\varepsilon,$$

with $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ in the above inequality, we get

$$\int_{\mathbb{R}^d\times W}\frac{|\gamma(t)-x-\int_0^t C_s^n(\gamma(s))ds|}{1+\|\gamma\|_\infty}\,d\eta \le ||V-C^n||_{L^1(\nu)}.$$

We have

$$\begin{aligned}&\int_{\mathbb{R}^d\times W}\frac{|\gamma(t)-x-\int_0^t V_s(\gamma(s))ds|}{1+\|\gamma\|_\infty}\,d\eta\\ &\le \int_0^t\int_{\mathbb{R}^d\times W}\frac{|V_s(\gamma(s))-C_s^n(\gamma(s))|}{1+|\gamma(s)|}\,d\eta\,ds + \int_0^t\int_{\mathbb{R}^d}\frac{|V_s-C_s^n|}{1+|x|}d\mu_s ds,\end{aligned}$$

which is less than $2||V-C^n||_{L^1(\nu)}$. Now letting $n\to+\infty$, we obtain that

$$\int_{\mathbb{R}^d\times W}\frac{|\gamma(t)-x-\int_0^t V_s(\gamma(s))ds|}{1+\|\gamma\|_\infty}d\eta = 0.$$

The result follows. □

Theorem 3.7 *(Ambrosio) The result in Theorem 3.6 subsists under the condition (3.3):*

$$\int_0^T\int_{\mathbb{R}^d}\frac{|V_t(x)|}{1+|x|}\,d\mu_t(x)dt < +\infty.$$

Proof. The technical hypothesis in the above theorem has been removed in [1]. □

Theorem 3.8 *Let V be a vector field satisfying the conditions:*

$$V\in L^1([0,T],W^{1,\alpha}_{loc}),\quad \mathrm{div}(V)\in L^1([0,T],L^\infty),$$

$$\frac{|V|}{1+|x|}\in L^1([0,T],L^1).$$

Then there exists a unique flow of measurable maps $X_t:\mathbb{R}^d\to\mathbb{R}^d$ such that for a.e. $x\in\mathbb{R}^d$,

$$X_t(x) = x+\int_0^t V_s(X_s(x))ds$$

and

$$(X_t)_*\lambda_d = k_t(x)\lambda_d.$$

Moreover

$$e^{-\int_0^T\|\mathrm{div}(V_s)\|_\infty ds} \le k_t(x) \le e^{\int_0^T\|\mathrm{div}(V_s)\|_\infty ds}.$$

Proof. Take $\mu_0 = \gamma_d \lambda_d$ the standard Gaussian measure: $\gamma_d \in L^1 \cap L^\infty$. By Theorem 2.8, there exists a unique $u \in L^\infty([0,T], L^1 \cap L^\infty)$ which solves the transport equation

$$\frac{\partial u_t}{\partial t} + V_t \cdot \nabla u_t + \mathrm{div}(V_t) u_t = 0.$$

Then $\mu_t := u_t \lambda_d$ solves the continuity equation (3.1). It is clear that the condition (3.3) holds. By Theorem 3.6, there is a probability measure η on $\mathbb{R}^d \times W$ under which

$$\gamma_t = x + \int_0^t V_s(\gamma_s) ds.$$

To complete the proof, we need the following lemma.

Lemma 3.9 *Let $\eta \in \mathcal{M}_+(\mathbb{R}^d \times W)$ be a probability measure such that*

$$\gamma(t) = x + \int_0^t V_s(\gamma(s)) ds, \quad \textit{holds } \eta\textit{-a.s.}$$

Denote by $e_t : \mathbb{R}^d \times W \to \mathbb{R}^d$ the evaluation map $e_t(x, \gamma) = \gamma(t)$ and $\mu_t^\eta = (e_t)_* \eta$. Suppose that

$$\int_0^T \int_{\mathbb{R}^d} \frac{|V_t(x)|}{1 + |x|} d\mu_t^\eta dt < +\infty.$$

Then $\{\mu_t^\eta\}$ satisfies the continuity equation

$$\frac{\partial \mu_t}{\partial t} + D_x \cdot (V_t \mu_t) = 0.$$

Proof. We claim that for η-a.s γ, $\int_0^T |V_s(\gamma(s))|\, ds < +\infty$; therefore $t \to \gamma(t)$ is absolutely continuous. In fact, the above condition can be rewritten in the form:

$$\int_0^T \mathbb{E}_\eta \Big(\frac{|V_t(\gamma(t))|}{1 + |\gamma(t)|} \Big) \, dt < +\infty.$$

It follows that for η-a.s γ, $\int_0^T \frac{|V_t(\gamma(t))|}{1+|\gamma(t)|} \, dt < +\infty$. For $R > 0$, consider the subset $A_R = \{(x, \gamma);\ ||\gamma||_\infty \le R\}$. Then $\mathbb{R}^d \times W = \cup_{R>0} A_R$. For $(x, \gamma) \in R$, we have

$$\int_0^T |V_t(\gamma(t))| \, dt \le (R+1) \int_0^T \frac{|V_t(\gamma(t))|}{1 + |\gamma(t)|} \, dt < +\infty.$$

Now $\gamma(t) = x + \int_0^t V_s(\gamma(s))\, ds$ yields the absolute continuity of γ. Let $\varphi \in C_c^\infty$; then

$$\begin{aligned} \frac{d}{dt} \langle \mu_t^\eta, \varphi \rangle &= \int_{\mathbb{R}^d \times W} \langle \nabla \varphi(\gamma(t)), V_t(\gamma(t)) \rangle d\eta \\ &= \langle \nabla \varphi \cdot V_t, \mu_t^\eta \rangle. \end{aligned}$$

The proof of the lemma is complete. □

End of the proof of Theorem 3.8 Let η_x be the conditional probability law of η, given $\gamma(0) = x$. Note that for $x \in \mathbb{R}^d$ given, η_x is concentrated on the subset of those γ such that $\gamma(t) = x + \int_0^t V_s(\gamma(s))\,ds$. We shall prove that η is supported by a graph: there exists a measurable map $X : \mathbb{R}^d \to W$ such that $\eta = (I, X)_*\mu_0$. If η is not supported by a graph, then there exists a $C \subset \mathbb{R}^d$ with $\mu_0(C) > 0$ and for each $x \in C$, η_x is not a Dirac mass on W_x. Then there exists $t_0 \in [0, T]$ and two disjoint Borel subsets $E, E' \subset \mathbb{R}^d$ such that

$$\eta_x\big(e_{t_0}^{-1}(E)\big) \cdot \eta_x\big(e_{t_0}^{-1}(E')\big) > 0 \quad \text{for } x \in C.$$

Note that we can choose C such that $\eta_x\big(e_{t_0}^{-1}(E)\big)$ and $\eta_x\big(e_{t_0}^{-1}(E')\big)$ are bounded below by a positive constant $\varepsilon_0 > 0$ for each $x \in C$. Define

$$\eta_x^1 = \mathbf{1}_C(x) \frac{\eta_x\Big(\mathbf{1}_{e_{t_0}^{-1}(E)\cap\cdot}\Big)}{\eta_x\big(e_{t_0}^{-1}(E)\big)}, \quad \eta_x^2 = \mathbf{1}_C(x) \frac{\eta_x\Big(\mathbf{1}_{e_{t_0}^{-1}(E')\cap\cdot}\Big)}{\eta_x\big(e_{t_0}^{-1}(E')\big)}.$$

Then η_x^1, η_x^2 are concentrated as η_x for $x \in C$ and we can check that the conditions in lemma 3.9 are satisfied. Therefore $\mu_t^{\eta^1}, \mu_t^{\eta^2}$ satisfy the continuity equation with the same initial measure $\mu_0^C = \mu_0(\mathbf{1}_C\cdot)$. Since η_x^1 and η_x^2 are absolutely continuous with respect to η_x, it is easy to see that $\mu_t^{\eta^1}, \mu_t^{\eta^1}$ admit density k_t^1 and k_t^2 in $L^\infty([0, T], L^1 \cap L^\infty)$. But in this class, the uniqueness holds due to Theorem 2.8. Then $\mu_t^{\eta^1} = \mu_t^{\eta^2}$, which is impossible. Therefore η is supported by the graph of a measurable map $X : \mathbb{R}^d \to W$; then

$$(X_t)_*\mu_0 = \mu_t = u_t\,\lambda_d.$$

Let $A \subset \mathbb{R}^d$ such that $\lambda_d(A) = 0$, then $(X_t)_*\mu_0(A) = 0$ or $\int \mathbf{1}_A(X_t)\,d\mu_0 = 0$ which implies that $\mathbf{1}_A(X_t) = 0$ a.e.; therefore $\int \mathbf{1}_A(X_t)\,dx = 0$. In other words, $(X_t)_*\lambda_d$ admits a density k_t with respect to λ_d. The estimates for k_t are similar to (1.2.9). □

4 Stochastic Differential Equations: Strong Solutions

Let $\sigma : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$ be a continuous function, taking values in the space of (d, m)-matrices and $V : \mathbb{R}^d \to \mathbb{R}^d$ a continuous function. Consider the following stochastic differential equation (abbreviated as SDE) on $\mathbb{R}^d$:

$$dX_t(\omega) = \sigma(X_t(\omega))dB_t(\omega) + V(X_t(\omega))dt, \quad X_0(\omega) = x \in \mathbb{R}^d, \tag{4.1}$$

where $t \to B_t(\omega)$ is a $\mathbb{R}^m$-valued standard Brownian motion. Now the situation is quite different since the Brownian motion is not unique (in the sense of paths), and the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ is not unique. How to understand (4.1)?

Definition 4.1 *(Strong Solution) If for any given Brownian motion $(B_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathbb{P})$, there exists a $(\mathcal{F}_t^B)_{t\geq 0}$-adapted process $(X_t(\omega))_{0\leq t<\tau_x}$ such that a.s.,*

$$X_t(\omega) = x + \int_0^t \sigma(X_s(\omega))dB_s + \int_0^t V(X_s(\omega))ds, \quad t < \tau_x(\omega), \tag{4.2}$$

where $\tau_x : \Omega \to [0, +\infty]$ is the life time, which is a $\mathcal{F}_\cdot^B$-stopping time.

Recall that $\mathcal{F}_t^B$ is the σ-field in Ω generated by $\{B_s(\cdot) : s \le t\}$; τ_x is a $\mathcal{F}_\cdot^B$-stopping time in the sense that

$$\{\omega : \tau_x(\omega) \le t\} \in \mathcal{F}_t^B, \quad \text{for all } t \ge 0.$$

If $\tau_x(\omega)$ is finite, then

$$\lim_{t \uparrow \tau_x(\omega)} |X_t(\omega)| = +\infty.$$

The uniqueness for strong solutions to (4.1) is understood as the "*pathwise uniqueness*": for two solutions $(X_t(\omega))_{0\le t<\tau_x^1}$ and $(Y_t(\omega))_{0\le t<\tau_x^2}$ to (4.1), such that $X_0 = Y_0$, then $\tau_x^1 = \tau_x^2$ and

$$\mathbb{P}(\omega : \ X_t(\omega) = Y_t(\omega) \text{ for all } t < \tau_x^1(\omega)) = 1.$$

The existence of strong solutions is very important in probabilistic modelisations.

Definition 4.2 *(Weak Solution) For the coefficients (σ, V) given, if there exists a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions on which are defined a $\mathcal{F}_t$-compatible Brownian motion $(B_t)_{t\ge0}$ and a $\mathcal{F}_t$-adapted process $(X_t)_{0<t<\tau_x}$ such that*

$$X_t(\omega) = X_0(\omega) + \int_0^t \sigma(X_s(\omega))dB_s(\omega) + \int_0^t V(X_s(\omega))ds, \quad t < \tau_x(\omega). \tag{4.3}$$

Here are some explanations on Definition 4.2.

(i) The usual conditions mean that the sub-σ-fields $\mathcal{F}_t$ are completed by null subsets in $\mathcal{F}$ and

$$\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}.$$

(ii) $(B_t)_{t\ge0}$ is compatible with $(\mathcal{F}_t)_{t\ge0}$ means $\mathcal{F}_t^B \subset \mathcal{F}_t$ and for any $s < t$, $B_t - B_s$ is independent of $\mathcal{F}_s$. It is equivalent to say that $\mathcal{F}_t^B \subset \mathcal{F}_t$ and

$$\mathbb{E}\big(e^{i\langle B_t - B_s, \xi\rangle}\big|\mathcal{F}_s\big) = e^{-(t-s)|\xi|^2/2}, \quad s < t. \tag{4.4}$$

(iii) X_t is measurable with respect to $\mathcal{F}_t$, not necessarily measurable with respect to $\mathcal{F}_t^B$. If this latter situation happens, then the weak solutions become strong ones.

The suitable notion of uniqueness for weak solutions is the "uniqueness in law". For the simplicity of exposition, we consider the case where $\tau_x = +\infty$. Then for $\omega \in \Omega$ given, $t \to X_t(\omega)$ is in $C([0, +\infty), \mathbb{R}^d)$. The law of the solution is the law of $\omega \to X_\cdot(\omega)$ on $C([0, +\infty), \mathbb{R}^d)$. The uniqueness in law means that if $\tilde{X}$ is another solution defined on another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$. Then

$$\text{law}(X_\cdot) = \text{law}(\tilde{X}_\cdot).$$

Construction of strong solutions:

First case. We suppose that the coefficients satisfy the global Lipschitz conditions:

$$||\sigma(x) - \sigma(y)|| \le C|x-y|, \quad |V(x) - V(y)| \le C|x-y|, \quad \forall\ x, y \in \mathbb{R}^d, \tag{4.5}$$

where $||\cdot||$ denotes the Hilbert-Schmidt norm for matrices: $||\sigma||^2 = \sum_{ij}(\sigma^{ij})^2$. In this case, the Picard iteration does work. More precisely, let $x \in \mathbb{R}^d$ be given. Define

$$\begin{aligned} X_0(t,x) &= x, \\ X_n(t,x) &= x + \int_0^t \sigma(X_{n-1}(s,x))dB_s + \int_0^t V(X_{n-1}(s,x))ds. \end{aligned}$$

The condition (4.5) implies that the coefficients have linear growths:

$$\|\sigma(x)\| \leq C_1(1+|x|), \quad |V(x)| \leq C_1(1+|x|). \tag{4.6}$$

We have

$$|X_n(t)|^2 \leq 3\left(|x|^2 + \left|\int_0^t \sigma(X_{n-1}(s))dB_s\right|^2 + \left|\int_0^t V(X_{n-1}(s))ds\right|^2\right). \tag{4.7}$$

Let $T > 0$. By Doob's maximal inequality,

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t \sigma(X_{n-1}(s))dB_s\right|^2\right) \leq 4\mathbb{E}\int_0^T \|\sigma(X_{n-1}(s))\|^2 ds \leq 8C_1^2\int_0^T \big(1+\mathbb{E}(|X_{n-1}(s)|^2)\big)ds,$$

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t V(X_{n-1}(s))ds\right|^2\right) \leq 2C_1^2 T\int_0^T \big(1+\mathbb{E}(|X_{n-1}(s)|^2)\big)ds.$$

According to (4.7) and by induction, we see that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X_n(t)|^2\right) < +\infty. \tag{4.8}$$

Now we compute

$$\begin{aligned} X_{n+1}(t) - X_n(t) &= \int_0^t \big(\sigma(X_n(s)) - \sigma(X_{n-1}(s))\big)dB_s + \int_0^t \big(V(X_n(s)) - V(X_{n-1}(s))\big)ds \\ &=: I_1(t) + I_2(t). \end{aligned}$$

Again by Doob's maximal martingale inequality,

$$\mathbb{E}\left(\sup_{0\leq s\leq t}|I_1(s)|^2\right) \leq 4\mathbb{E}\int_0^t \|\sigma(X_n(s)) - \sigma(X_{n-1}(s))\|^2 ds \leq 4C^2\int_0^t \mathbb{E}\big(|X_n(s) - X_{n-1}(s)|^2\big)ds.$$

In the same way,

$$\mathbb{E}\left(\sup_{0\leq s\leq t}|I_2(s)|^2\right) \leq TC^2\int_0^t \mathbb{E}\big(|X_n(s) - X_{n-1}(s)|^2\big)ds.$$

Therefore

$$\mathbb{E}\left(\sup_{0\leq s\leq t}|X_{n+1}(s) - X_n(s)|^2\right) \leq 2C^2(T+4)\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s}|X_n(u) - X_{n-1}(u)|^2\right)ds.$$

By induction,

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|X_{n+1}(s) - X_n(s)|^2\right) \leq K\frac{(2C^2(T+4))^n}{n!}T^n.$$

Hence

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|X_{n+1}(t) - X_n(t)| > 2^{-n}\right) \leq K\frac{(8C^2(T+4))^n}{n!}.$$

By Borel-Cantelli lemma, a.s.

$$\sup_{0\leq t\leq T}|X_{n+1}(t) - X_n(t)| \leq 2^{-n} \quad \text{as } n\uparrow+\infty.$$

It follows that the series

$$X_t(x,\omega) := x + \sum_{n=0}^{+\infty} \big(X_{n+1}(t) - X_n(t)\big)$$

converges uniformly with respect to $t \in [0,T]$. Now remark that

$$\mathbb{E}\bigg|\int_0^t \sigma(X_n(s))dB_s - \int_0^t \sigma(X(s))dB_s\bigg|^2 \le C^2\mathbb{E}\bigg(\int_0^t |X_n(s) - X_s|^2 ds\bigg) \to 0 \quad \text{as } n \to +\infty.$$

So letting $n \to +\infty$ in

$$X_{n+1}(t) = x + \int_0^t \sigma(X_n(s))dB_s + \int_0^t V(X_n(s))ds,$$

we get

$$X_t(x) = x + \int_0^t \sigma(X_s(x))dB_s + \int_0^t V(X_s(x))ds.$$

□

A comment on the lifetime τ_x. By the above construction, we see that for $x \in \mathbb{R}^d$, $\tau_x = +\infty$ a.s. More precisely, let

$$A = \{(x,\omega) \in \mathbb{R}^d \times \Omega : \tau_x(\omega) = +\infty\}.$$

Let ν be a Borel probability measure on $\mathbb{R}^d$, then

$$(\nu \otimes \mathbb{P})(A) = 1.$$

By Fubini theorem,

$$\int_\Omega \bigg[\int_{\mathbb{R}^d} \mathbf{1}_A(x,\omega)d\nu(x)\bigg] d\mathbb{P}(\omega) = 1.$$

It follows that there exists $\Omega_0 \subset \Omega$ with full probability, such that for each $\omega \in \Omega_0$, $\tau_x(\omega) = +\infty$ for ν-a.e. x. By limit procedure, we see that $(x,\omega) \to X_t(x,\omega)$ is measurable with respect to $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t}^{\nu\otimes\mathbb{P}}$.

A new definition of strong solution: Let $W(\mathbb{R}^d) = C([0,+\infty),\mathbb{R}^d)$ and $W_0(\mathbb{R}^d) = \{w \in W(\mathbb{R}^d) : w(0) = 0\}$. Let μ_W be the law of the Brownian motion on $W_0(\mathbb{R}^m)$. We say that the SDE (4.1) has a strong solution if there exists $F : \mathbb{R}^d \times W_0(\mathbb{R}^m) \to W(\mathbb{R}^d)$ such that

(i) for any Borel probability measure ν on $\mathbb{R}^d$, F is measurable with respect to

$$\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0(\mathbb{R}^m))}^{\nu\otimes\mu_W};$$

(ii) for any t, x given, $w \to F(x,w)(t)$ is $\mathcal{F}_t^W$-measurable, where $\mathcal{F}_t^W = \sigma(w(s) : s \le t)$, and $X_t(w) = F(X_0(w), B)$ is a strong solution to (4.1) in the sense of Definition 4.1, where X_0 is random variable independent of B.

Second case. Now we will give another example, for which the strong solution exists. Suppose that σ and V are bounded and

$$\|\sigma(x) - \sigma(y)\|^2 \le C|x-y|^2 \log\frac{1}{|x-y|}, \quad |V(x) - V(y)| \le C|x-y| \log\frac{1}{|x-y|} \tag{4.9}$$

for any $|x-y| \leq \delta_0$. In this case, the Euler approximation does work. Let $n \geq 1$, denote

$$\Phi_n(t) = k2^{-n} \quad \text{for } t \in [k2^{-n}, (k+1)2^{-n}),\ k \geq 0.$$

Define $X_n(0) = x$ and for $t \in [k2^{-n}, (k+1)2^{-n})$,

$$X_n(t) = X_n(k2^{-n}) + \sigma(X_n(k2^{-n}))(B_t - B_{k2^{-n}}) + V(X_n(k2^{-n}))(t - k2^{-n}) \in \mathcal{F}_t^B.$$

Using Φ_n, $X_n(t)$ can be expressed by

$$X_n(t) = x + \int_0^t \sigma\big(X_n(\Phi_n(s))\big)dB_s + \int_0^t V\big(X_n(\Phi_n(s))\big)ds.$$

Then we can prove that a.s., $X_n(t)$ converges to X_t uniformly with respect to $t \in [0,T]$. Therefore, in this case, the strong solution exists. For a proof, we refer to [4]. □

Theorem 4.3 *Under* (4.9)*, the pathwise uniqueness holds.*

Proof. Let X_t and Y_t be two solutions to (4.1) starting from the same point. Set $\eta_t = X_t - Y_t$ and $\xi_t = |\eta_t|^2$. We have

$$d\eta_t = (\sigma(X_t) - \sigma(Y_t))dB_t + (V(X_t) - V(Y_t))dt.$$

In coordinate forms, for $i = 1, \cdots, d$,

$$d\eta_t^i = \sum_{j=1}^m (\sigma^{ij}(X_t) - \sigma^{ij}(Y_t))dB_t^j + (V^i(X_t) - V^i(Y_t))dt.$$

Itô stochastic contraction of $d\eta_t$ is given by

$$\sum_{i=1}^d d\eta_t^i \cdot d\eta_t^i = \sum_{i,j} (\sigma^{ij}(X_t) - \sigma^{ij}(Y_t))^2 dt = ||\sigma(X_t) - \sigma(Y_t)||^2\, dt.$$

Now by Itô formula,

$$\begin{aligned} d\xi_t &= 2\langle \eta_t, d\eta_t\rangle + d\eta_t \cdot d\eta_t \\ &= 2\langle \eta_t, (\sigma(X_t) - \sigma(Y_t))dB_t\rangle + 2\langle \eta_t, V(X_t) - V(Y_t)\rangle dt + \|\sigma(X_t) - \sigma(Y_t)\|^2 dt \\ &= 2\langle (\sigma(X_t) - \sigma(Y_t))^*\eta_t, dB_t\rangle + 2\langle \eta_t, V(X_t) - V(Y_t)\rangle dt + \|\sigma(X_t) - \sigma(Y_t)\|^2 dt. \end{aligned}$$

We see that

$$d\xi_t \cdot d\xi_t = 4\Big|(\sigma(X_t) - \sigma(Y_t))^*\eta_t\Big|^2 dt.$$

Let $\varepsilon > 0$. Define $\Psi_\varepsilon(\xi) = \int_0^t \frac{ds}{s\log\frac{1}{s} + \varepsilon}$, $\Phi_\varepsilon(\xi) = e^{\Psi_\varepsilon(\xi)}$. Then

$$\begin{aligned} \Phi_\varepsilon' &= \frac{\Phi_\varepsilon}{\xi\log\frac{1}{\xi} + \varepsilon}, \\ \Phi_\varepsilon'' &= \frac{\Phi_\varepsilon'\big(\xi\log\frac{1}{\xi} + \varepsilon\big) + \Phi_\varepsilon \cdot (\log\xi + 1)}{\big(\xi\log\frac{1}{\xi} + \varepsilon\big)^2} = \frac{\Phi_\varepsilon \cdot (\log\xi + 2)}{\big(\xi\log\frac{1}{\xi} + \varepsilon\big)^2} \leq 0 \quad \text{if } \xi \leq e^{-2}. \end{aligned}$$

Define

$$\tau = \inf\{t > 0 : \xi_t \geq e^{-2} \wedge \delta_0^2\}.$$

By Itô formula,

$$\begin{aligned}
\Phi_\varepsilon(\xi_{t\wedge\tau}) &= 1 + \int_0^{t\wedge\tau} \Phi_\varepsilon'(\xi_s)d\xi_s + \frac{1}{2}\int_0^{t\wedge\tau} \Phi_\varepsilon''(\xi_s)d\xi_s \cdot d\xi_s \\
&= 1 + 2\int_0^{t\wedge\tau} \Phi_\varepsilon'(\xi_s)\langle(\sigma(X_s) - \sigma(Y_s))^*\eta_s, dB_s\rangle + 2\int_0^{t\wedge\tau} \Phi_\varepsilon'(\xi_s)\langle\eta_s, V(X_s) - V(Y_s)\rangle ds \\
&\quad + \int_0^{t\wedge\tau} \Phi_\varepsilon'(\xi_s)\|\sigma(X_s) - \sigma(Y_s)\|^2 ds + \frac{1}{2}\int_0^{t\wedge\tau} \Phi_\varepsilon''(\xi_s)d\xi_s \cdot d\xi_s.
\end{aligned}$$

Taking the expectation,

$$\mathbb{E}\big(\Phi_\varepsilon(\xi_{t\wedge\tau})\big) \leq 1 + 2\mathbb{E}\int_0^{t\wedge\tau} \frac{\Phi_\varepsilon \cdot C\xi_s \log\frac{1}{\xi_s}}{\xi_s \log\frac{1}{\xi_s} + \varepsilon} ds \leq 1 + 2C\int_0^t \mathbb{E}\big(\Phi_\varepsilon(\xi_{t\wedge\tau})\big)ds.$$

By Gronwall lemma,

$$\mathbb{E}\big(\Phi_\varepsilon(\xi_{t\wedge\tau})\big) \leq e^{2Ct}.$$

Letting $\varepsilon \downarrow 0$, we get $\xi_{t\wedge\tau} = 0$ a.s. If $\mathbb{P}(\tau < T) > 0$, then by continuity of the paths, we get

$$\xi_\tau = 0 \quad \text{on } \{\tau < T\}.$$

But $\xi_\tau = \delta_0^2 \wedge e^{-2}$. The contradiction implies that $\tau \geq T$ a.s., and hence $\xi_t = 0$ a.s. □

5 Stochastic Flow of Homeomorphisms

In contrast to ODE, even under the global Lipschitz condition:

$$\|\sigma(x) - \sigma(y)\| \leq C|x - y|, \quad |V(x) - V(y)| \leq C|x - y|, \tag{5.1}$$

for a study of the dependence $x \to X_t(x, \omega)$, we need the following Kolmogorov modification theorem, one of the fundamental tools in probability theory:

Theorem 5.1 *(Kolmogorov) Let $\{X_t : t \in [0,1]^m\}$ be a family of $\mathbb{R}^d$-valued random variables. Suppose there exist constants $\gamma \geq 1$, C, $\delta > 0$ such that*

$$\mathbb{E}\big(|X_t - X_s|_{\mathbb{R}^d}^\gamma\big) \leq C|t - s|_{\mathbb{R}^m}^{m+\delta}. \tag{5.2}$$

Then X admits a continuous version $\tilde{X}_t$, satisfying

$$\mathbb{E}\left[\sup_{s\neq t}\left(\frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\alpha}\right)^\gamma\right] < +\infty \quad \text{for any } \alpha \in \left(0, \frac{\delta}{\gamma}\right). \tag{5.3}$$

In particular, there exists $M \in L^\gamma(\Omega)$ such that

$$|\tilde{X}_t - \tilde{X}_s| \leq M|t - s|^\alpha, \quad \forall\, t, s \in [0,1]^m. \tag{5.4}$$

Remark 5.2 (1) $\tilde{X}$ is a version of X means that for each $t \in [0,1]^m$, $X_t = \tilde{X}_t$ a.s.

(2) X_t could take values in a Banach space, but it is crucial that the exponent of $|t-s|_{\mathbb{R}^m}$ is strictly bigger than m, namely $\delta > 0$.

(3) Let $\{X_t^n : t \in [0,1]^m\}$ be a sequence of such family satisfying (5.2), but with C independent of n. Then there exist $M_n \in L^\gamma$ bounded in L^γ such that

$$|\tilde{X}_t^n - \tilde{X}_s^n| \le M_n \, |t-s|_{\mathbb{R}^m}^{\alpha}, \quad \forall \, t,s \in [0,1]^m.$$

□

The proof of Theorem 5.1 can be found in any textbook of probability theory. We refer to [6].

Let $X_t(x,\omega)$ be the solution to

$$dX_t(\omega) = \sigma(X_t(\omega))dB_t + V(X_t(\omega))dt, \quad X_0(\omega) = x. \tag{5.5}$$

Theorem 5.3 *Under the global Lipschitz condition* (5.1), *the solution to* (5.5) *admits a continuous version* $\tilde{X}$, *such that a.s.,* $(t,x) \to \tilde{X}_t(x,\omega)$ *is continuous and for each* $x \in \mathbb{R}^d$,

$$\mathbb{P}\{\omega : \forall \, t \ge 0, \; X_t(x,\omega) = \tilde{X}_t(x,\omega)\} = 1. \tag{5.6}$$

Proof. Let $R > 0$, $T > 0$ and set $I = [0,T] \times [-R,R]^d$. Consider $\eta_t = X_t(x) - X_t(y)$. We have

$$d\eta_t = \big(\sigma(X_t(x)) - \sigma(X_t(y))\big)dB_t + \big(V(X_t(x)) - V(X_t(y))\big)dt.$$

The Itô stochastic contraction of $d\eta_t$ is given by

$$d\eta_t \cdot d\eta_t = \|\sigma(X_t(x)) - \sigma(X_t(y))\|^2 dt.$$

Let $\xi_t = |\eta_t|^2$. Then by Itô formula,

$$\begin{aligned} d\xi_t &= 2\langle \eta_t, d\eta_t \rangle + d\eta_t \cdot d\eta_t \\ &= 2\big\langle \eta_t, \big(\sigma(X_t(x)) - \sigma(X_t(y))\big)dB_t \big\rangle + 2\langle \eta_t, V(X_t(x)) - V(X_t(y))\rangle dt \\ &\quad + \|\sigma(X_t(x)) - \sigma(X_t(y))\|^2 dt. \end{aligned}$$

The Itô stochastic contraction of $d\xi_t$ is

$$d\xi_t \cdot d\xi_t = 4\big|\big(\sigma(X_t(x)) - \sigma(X_t(y))\big)^* \eta_t\big|^2 dt.$$

Let $p \ge 2$. By Itô formula,

$$\begin{aligned} d\xi_t^p &= p\xi_t^{p-1} d\xi_t + \frac{1}{2}p(p-1)\xi_t^{p-2} d\xi_t \cdot d\xi_t \\ &= 2p\xi_t^{p-1}\big\langle \eta_t, \big(\sigma(X_t(x)) - \sigma(X_t(y))\big)dB_t \big\rangle + 2p\xi_t^{p-1}\langle \eta_t, V(X_t(x)) - V(X_t(y))\rangle dt \\ &\quad + p\xi_t^{p-1}\|\sigma(X_t(x)) - \sigma(X_t(y))\|^2 dt + 2p(p-1)\xi_t^{p-2}\big|\big(\sigma(X_t(x)) - \sigma(X_t(y))\big)^* \eta_t\big|^2 dt \\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned}$$

By the Lipschitz conditions,

$$\begin{aligned} |I_2(t)| &\le 2p\xi_t^{p-1} \cdot C|\eta_t|^2 = 2pC\xi_t^p, \\ |I_3(t)| &\le p\xi_t^{p-1} \cdot C^2 \xi_t = pC^2\xi_t^p, \\ |I_4(t)| &\le 2p(p-1)\xi_t^{p-2}|\eta_t|^2 C^2 |\eta_t|^2 = 2p(p-1)C^2\xi_t^p. \end{aligned}$$

Therefore for some constant $C_p > 0$, we have

$$\mathbb{E}(\xi_t^p) \leq |x-y|^{2p} + C_p \int_0^t \mathbb{E}(\xi_s^p) ds.$$

By Gronwall lemma,

$$\mathbb{E}(\xi_t^p) \leq |x-y|^{2p} e^{C_p T} \quad \text{for any } t \leq T,$$

or

$$\mathbb{E}(|X_t(x) - X_t(y)|^{2p}) \leq |x-y|^{2p} e^{C_p T}. \tag{5.7}$$

Next, there exists a $C_p > 0$ such that

$$|X_t(x)|^{2p} \leq C_p \left(|x|^{2p} + \left| \int_0^t \sigma(X_s(x)) dB_s \right|^{2p} + \left| \int_0^t V(X_s(x)) ds \right|^{2p} \right). \tag{5.8}$$

By Burkhölder inequality, we have

$$\begin{aligned} \mathbb{E}\left(\sup_{0\leq t\leq T} \left| \int_0^t \sigma(X_s(x)) dB_s \right|^{2p} \right) &\leq C_p \mathbb{E}\left(\int_0^t \|\sigma(X_s(x))\|^2 ds \right)^p \\ &\leq \tilde{C}_{p,T} \left[1 + \mathbb{E}\left(\sup_{0\leq t\leq T} |X_t(x)|^{2p} \right) \right]. \end{aligned}$$

If we denote by $u_t = \mathbb{E}\big(\sup_{0\leq s\leq t} |X_s(x)|^{2p}\big)$, then by (5.8),

$$u_t \leq C_{p,T} \left(|x|^{2p} + \int_0^t u_s ds \right).$$

Again by Gronwall lemma, we get

$$u_t \leq C_{p,T} |x|^{2p} e^{t C_{p,T}},$$

or

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |X_t(x)|^{2p} \right) \leq C_{p,T} |x|^{2p} e^{T C_{p,T}}. \tag{5.9}$$

Now for $t > s$,

$$X_t - X_s = \int_s^t \sigma(X_r) dB_r + \int_s^t V(X_r) dr.$$

Again by Bürkhölder inequality:

$$\begin{aligned} \mathbb{E}\left(\left| \int_s^t \sigma(X_r) dB_r \right|^{2p} \right) &\leq C_p \mathbb{E}\left(\int_s^t \|\sigma(X_r)\|^2 dr \right)^p \leq \tilde{C}_p \mathbb{E}\left[\int_s^t \left(1 + \sup_{0\leq r\leq T} |X_r|^2 \right) dr \right]^p \\ &\leq \tilde{C}_p \left[1 + \mathbb{E}\left(\sup_{0\leq t\leq T} |X_t|^{2p} \right) \right] (t-s)^p. \end{aligned}$$

Finally we get

$$\mathbb{E}\big(|X_t(x) - X_s(x)|^{2p}\big) \leq C_{p,T} |t-s|^p.$$

Combining with (5.7), we have

$$\mathbb{E}\big(|X_t(x) - X_s(y)|^{2p}\big) \leq C_{p,T}\big(|t-s|^p + |x-y|^{2p}\big). \tag{5.10}$$

Now take $p > d+1$, we can apply Kolmogorov's modification theorem: there exists a continuous version $\tilde{X}$ such that for given $(t,x) \in [0,T] \times [-R,R]^d$, a.s. $X_t(x,\omega) = \tilde{X}_t(x,\omega)$. Since the two processes are continuous with respect to t, we obtain for given $x \in \mathbb{R}^d$, a.s.

$$X_t(x,\omega) = \tilde{X}_t(x,\omega) \quad \text{for all } t \geq 0.$$

□

Remark 5.4 *By* (5.7)*, we see that $x \to \tilde{X}_t(x,\omega)$ is $(1-\varepsilon)$-Hölder continuous: there exists a loss of regularity with respect to the coefficients.* □

The main result of this section is

Theorem 5.5 *There exists a full measure subset $\Omega_0 \subset \Omega$ such that for $\omega \in \Omega_0$ and any $t \geq 0$, $x \to \tilde{X}_t(x,\omega)$ is a global homeomorphism of $\mathbb{R}^d$.*

Proof. Step 1. Let $x \neq y$ be given, for $\alpha < 0$, we estimate $\mathbb{E}\big(|X_t(x) - X_t(y)|^{2\alpha}\big)$.

Take $\varepsilon \in (0, |x-y|^2)$. Set $\eta_t = X_t(x) - X_t(y)$ and $\xi_t = |\eta_t|^2$. We have $\xi_0 = |x-y|^2 > \varepsilon$. Define the stopping time

$$\tau_\varepsilon = \inf\{t > 0 : \xi_t \leq \varepsilon\}.$$

Then $\xi_{t\wedge\tau_\varepsilon} \geq \varepsilon$. By Itô formula,

$$\begin{aligned}
\xi^\alpha_{t\wedge\tau_\varepsilon} &= |x-y|^{2\alpha} + \alpha \int_0^{t\wedge\tau_\varepsilon} \xi_s^{\alpha-1} d\xi_s + \frac{1}{2}\alpha(\alpha-1)\int_0^{t\wedge\tau_\varepsilon} \xi_s^{\alpha-2} d\xi_s \cdot d\xi_s \\
&\leq |x-y|^{2\alpha} + M_{t\wedge\tau_\varepsilon} + 2\alpha \int_0^{t\wedge\tau_\varepsilon} \xi_s^{\alpha-1} \langle \eta_s, V(X_s(x)) - V(X_s(y))\rangle ds \\
&\quad + \frac{1}{2}\alpha(\alpha-1)\int_0^{t\wedge\tau_\varepsilon} \xi_s^{\alpha-2} d\xi_s \cdot d\xi_s \\
&=: |x-y|^{2\alpha} + I_1 + I_2 + I_3.
\end{aligned}$$

By Lipschitz condition,

$$|I_2| \leq 2|\alpha| C \int_0^{t\wedge\tau_\varepsilon} \xi_s^\alpha ds.$$

In the same way, there exists a constant $C_\alpha > 0$ such that

$$\mathbb{E}\big(\xi^\alpha_{t\wedge\tau_\varepsilon}\big) \leq |x-y|^{2\alpha} + C_\alpha \int_0^t \mathbb{E}\big(\xi^\alpha_{s\wedge\tau_\varepsilon}\big) ds.$$

It follows that

$$\mathbb{E}\big(\xi^\alpha_{t\wedge\tau_\varepsilon}\big) \leq |x-y|^{2\alpha} e^{tC_\alpha}. \tag{5.11}$$

Let $\tau_0 = \inf\{t > 0 : |X_t(x) - X_t(y)| = 0\}$, we see that

$$\tau_\varepsilon \uparrow \tau_0 \quad \text{as} \quad \varepsilon \downarrow 0.$$

Then letting $\varepsilon \downarrow 0$ in (5.11), we get

$$\mathbb{E}\big(\xi^{\alpha}_{t\wedge\tau_0}\big) \le |x-y|^{2\alpha} e^{tC_\alpha}.$$

If $\mathbb{P}(\tau_0 < +\infty) > 0$, then $\mathbb{P}(\tau_0 \le T) > 0$ for some $T > 0$. Therefore

$$+\infty = \mathbb{E}\big(\mathbf{1}_{\{\tau_0\le T\}}\xi^{\alpha}_{T\wedge\tau_0}\big) \le \mathbb{E}\big(\xi^{\alpha}_{T\wedge\tau_0}\big) \le |x-y|^{2\alpha} e^{TC_\alpha} < +\infty.$$

The contradiction implies that $\tau_0 = +\infty$ a.s., in other words,

$$\text{a.s.} \quad X_t(x) \ne X_t(y) \quad \text{for all } t \ge 0.$$

Remark 5.6 *Here the "a.s." is dependent of the given $x \ne y$.* □

Step 2. Let $\delta > 0$ be given. Set

$$\triangle^R_\delta = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \ge \delta, |x| \le R, |y| \le R\}.$$

Let $X_t(x)$ be the continuous version. Define

$$\eta_t(x,y) = |X_t(x) - X_t(y)|^{-1}.$$

For $p \ge 2$, (x,y) and $(\tilde{x},\tilde{y}) \in \triangle^R_\delta$,

$$\begin{aligned}|\eta_t(x,y) - \eta_s(\tilde{x},\tilde{y})| &\le \eta_t(x,y)\eta_s(\tilde{x},\tilde{y})\big||X_s(\tilde{x}) - X_s(\tilde{y})| - |X_t(x) - X_t(y)|\big| \\ &\le \eta_t(x,y)\eta_s(\tilde{x},\tilde{y})\big(|X_s(\tilde{x}) - X_t(x)| + |X_s(\tilde{y}) - X_t(y)|\big).\end{aligned}$$

By (5.11),

$$\mathbb{E}\big(\eta_t(x,y)^{4p}\big) \le e^{C_{-p}T}\delta^{-4p}, \quad \mathbb{E}\big(\eta_s(\tilde{x},\tilde{y})^{4p}\big) \le e^{C_{-p}T}\delta^{-4p}.$$

Combining with (5.10), we have

$$\mathbb{E}\big(|\eta_t(x,y) - \eta_s(\tilde{x},\tilde{y})|^p\big) \le C_{p,T,\delta}\big(|t-s|^{\frac{p}{2}} + |\tilde{x}-x|^p + |\tilde{y}-y|^p\big).$$

Taking $\frac{p}{2} > 2d+1$, again by Kolmogorov's modification theorem, $\eta_t(x,y)$ admits a continuous version $\tilde{\eta}_t(x,y) : (t,x,y) \to \tilde{\eta}_t(x,y)$ on $[0,T] \times \triangle^R_\delta$. Since $\delta > 0$, $R > 0$ are arbitrary, we have

$$\text{a.s.,} \quad (t,x,y) \to \tilde{\eta}_t(x,y) \text{ is continuous on } [0,T] \times \triangle_0,$$

where $\triangle_0 = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x \ne y\}$.

Let D be a dense countable subset of $[0,T] \times \triangle_0$. Then there exists a subset $\Omega_0 \subset \Omega$ of full measure such that for every $\omega \in \Omega_0$, $(t,x,y) \in D$, $\eta_t(x,y) = \tilde{\eta}_t(x,y)$, or

$$|X_t(x) - X_t(y)| = (\tilde{\eta}_t(x,y))^{-1}.$$

By continuity, it holds for all $(t,x,y) \in [0,T] \times \triangle_0$; especially $|X_t(x) - X_t(y)| = (\tilde{\eta}_t(x,y))^{-1} \ne 0$. In conclusion, we get a full measure subset Ω_0 (independent of x, y), such that for each $\omega \in \Omega_0$, $\forall\, t \ge 0$, $\forall\, x \ne y$, we have $X_t(x) \ne X_t(y)$.

Step 3 (Surjectivity): By considering $\hat{x} = \frac{x}{|x|^2}$ and

$$\eta_t(\hat{x}) = \begin{cases} (1 + |X_t(x)|)^{-1}, & \text{if } \hat{x} \ne 0; \\ 0, & \text{if } \hat{x} = 0, \end{cases}$$

the same machinery shows that $x \to X_t(x)$ is continuous at ∞. Set $\overline{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ which is homeomorphic to S^d. Then almost surely for all $t \geq 0$, $x \to X_t(x,\omega)$ is continuous from $\overline{\mathbb{R}}^d$ onto $\overline{\mathbb{R}}^d$. But $X_0(\cdot,\omega) = Id$, therefore $X_t(\cdot,\omega)$ is homotopic to Id. By a result from General Topology, $X_t(\overline{\mathbb{R}}^d) = \overline{\mathbb{R}}^d$. In particular,

$$X_t(\mathbb{R}^d) = \mathbb{R}^d.$$

□

Due to Remark 5.4, in order that for a.s $\omega \in \Omega$, $x \to X_t(x,\omega)$ is C^2, we have to suppose that $\sigma \in C_b^{2+\delta}, V \in C_b^{2+\delta}$, the space of bounded functions having bounded first and second order derivatives and the second order derivative being δ-Hölder continuous; for more detail, we refer to Kunita [8]. In what follows, we assume that the coefficients satisfy these conditions. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, consider $(P_t\varphi)(x) = \mathbb{E}(\varphi(X_t(x,\cdot))$. Then $(t,x) \to (P_t\varphi)(x)$ is in C_b^2. By Itô formula,

$$\mathbb{E}\big(\varphi(X_t(x))\big) = \varphi(x) + \int_0^t \mathbb{E}\big((L\varphi)(X_s(x))\big)ds,$$

where

$$(L\varphi)(x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x) + \sum_{i=1}^d V_i(x)\frac{\partial\varphi}{\partial x_i}(x)$$

with $a = \sigma\sigma^*$. Now let μ_0 be a probability measure on $\mathbb{R}^d$ and $\mathbb{P}_{\mu_0}$ the diffusion distribution on $W = C([0,T],\mathbb{R}^d)$ defined by $\mathbb{P}_{\mu_0} = \int_{\mathbb{R}^d} \mathbb{P}_x d\mu_0(x)$, where $\mathbb{P}_x$ is the law of $\omega \to X_\cdot(x,\omega)$. Define $\mu_t \in \mathcal{P}(\mathbb{R}^d)$ by

$$\int_{\mathbb{R}^d} \varphi\, d\mu_t = \int_W \varphi(\gamma(t))d\mathbb{P}_{\mu_0} = \int_{\mathbb{R}^d} \mathbb{E}\big(\varphi(X_t(x,\omega))\big)d\mu_0(x), \quad \text{for all } \varphi \in C_b(\mathbb{R}^d).$$

Therefore for $\varphi \in C_b^2(\mathbb{R}^d)$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi\, d\mu_t = \int_{\mathbb{R}^d} \mathbb{E}\big((L\varphi)(X_t(x))\big)d\mu_0(x) = \int_{\mathbb{R}^d} L\varphi\, d\mu_t. \tag{5.12}$$

Definition 5.7 *Let $(\mu_t)_{t\in[0,T]}$ be a family of probability measures on $\mathbb{R}^d$. We say that (μ_t) satisfies the following Fokker-Planck equation:*

$$\frac{\partial \mu_t}{\partial t} = L^*\mu_t, \quad \mu|_{t=0} = \mu_0, \tag{5.13}$$

if for $\psi \in C_c^\infty((0,T)\times\mathbb{R}^d)$,

$$-\int \frac{\partial\psi}{\partial t}\, d\mu_t dt = \int L\psi\, d\mu_t dt. \tag{5.14}$$

As what was done in Theorem 3.2, (μ_t) admits a version $(\tilde{\mu}_t)$ such that $t \to \tilde{\mu}_t$ is weakly continuous. Therefore the equation (5.14) holds for $\psi \in C_b^2([0,T]\times\mathbb{R}^d)$.

Theorem 5.8 *Suppose that $a \in C_b^{2+\delta}$ and $V \in C_b^{2+\delta}$. Then the Fokker-Planck equation* (5.13) *admits a unique solution.*

Proof. By linearity, it is sufficient to prove that $\mu_t = 0$ for all $t \in [0,T]$ if $\mu_0 = 0$. Fix $\varphi \in C_c^\infty(\mathbb{R}^d)$, $t_0 \in [0,T]$. Then the following backward PDE admits a unique solution $f \in C_b^2$:

$$\begin{cases} \frac{\partial f}{\partial t} + Lf = 0 & \text{in } [0,t_0] \times \mathbb{R}^d, \\ f(t_0, \cdot) = \varphi. \end{cases} \tag{5.15}$$

In fact, $f(t,x) = (P_{t_0-t}\varphi)(x)$ is the candidate. Let $\alpha \in C_c^\infty((0,T))$. By (5.14),

$$-\int \frac{\partial}{\partial t}(\alpha f)\, d\mu_t dt = \int \alpha\, Lf\, d\mu_t dt,$$

or

$$-\int_0^T \alpha'(t)\Big(\int_{\mathbb{R}^d} f(t,x) d\mu_t(x)\Big) dt = \int_0^T \alpha(t)\Big(\int_{\mathbb{R}^d} (\frac{\partial f}{\partial t} + Lf)\, d\mu_t\Big) dt.$$

It follows that

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x) d\mu_t(x) = \int_{\mathbb{R}^d} (\frac{\partial f}{\partial t} + Lf)\, d\mu_t = 0.$$

Therefore

$$0 = \int_{\mathbb{R}^d} f(0,x) d\mu_0(x) = \int_{\mathbb{R}^d} f(t_0,x) d\mu_{t_0}(x) = \int_{\mathbb{R}^d} \varphi(x) d\mu_{t_0}(x).$$

It follows that $\mu_{t_0} = 0$. As t_0 is arbitrary, we get the result. □

6 Stochastic Differential Equations: Weak Solutions

Let $\sigma : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$ and $V : \mathbb{R}^d \to \mathbb{R}^d$ be bounded Borel functions. Suppose that (X_t, B_t) solves

$$dX_t(\omega) = \sigma(X_t(\omega)) dB_t + V(X_t(\omega)) dt. \tag{6.1}$$

Then by Itô formula, for any $f \in C_c^2(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t (Lf)(X_s) ds = \int_0^t \langle \sigma^*(X_s)\nabla f(X_s), dB_s \rangle$$

is a L^2-martingale, where

$$(Lf)(x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d V_i(x)\frac{\partial f}{\partial x_i}(x), \quad a = \sigma\sigma^*. \tag{6.2}$$

In general, for a given operator L such as in (6.2), we say that a continuous adapted process $(X_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a solution to L-martingale problem if for any $f \in C_c^2(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t (Lf)(X_s) ds$$

is a continuous martingale. If furthermore the matrix a admits a decomposition $a = \sigma\sigma^*$ with $\sigma : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$, then there is a Brownian motion compatible with $(\mathcal{F}_t)$ such that X_t solves (6.1). This topic is well treated in Stroock-Varadhan's book [11]. In what follows, we will construct a weak solution.

Theorem 6.1 *Let σ, V be bounded continuous and $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ with compact support. Then the SDE* (6.1) *admits a weak solution (X_t, B_t) such that $\mu_0 = law(X_0)$*

Proof. First there exists a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ on which are defined a compatible Brownian motion (B_t) and a random variable $\xi_0 \in \mathcal{F}_0$ with $\text{law}(\xi_0) = \mu_0$. Next for $n \geq 1$, define $X_n(0) = \xi$ and for $t \in [k2^{-n}, (k+1)2^{-n})$,

$$X_n(t) = X_n(k2^{-n}) + \sigma\big(X_n(k2^{-n})\big)(B_t - B_{k2^{-n}}) + V(X_n(k2^{-n}))(t - k2^{-n}).$$

Using $\Phi_n(t) = k2^{-n}$ for $t \in [k2^{-n}, (k+1)2^{-n})$, we express $X_n(t)$ as

$$X_n(t) = \xi + \int_0^t \sigma\big(X_n(\Phi_n(s))\big)dB_s + \int_0^t V\big(X_n(\Phi_n(s))\big)ds.$$

As σ and V are bounded, for $T > 0$ fixed, by Bürkhölder inequality,

$$\sup_n \mathbb{E}\Big(\sup_{0\leq t\leq T} |X_n(t)|^{2p}\Big) < +\infty, \quad \sup_n \mathbb{E}\big(|X_n(t) - X_n(s)|^{2p}\big) \leq C_{p,T}|t-s|^p.$$

By Kolmogorov's modification theorem, there exists $M_n \in L^{2p}$ bounded in L^{2p} such that

$$|X_n(t) - X_n(s)| \leq M_n|t-s|^\alpha, \quad \alpha < \frac{p-1}{2p}. \tag{6.3}$$

Let

$$K_R = \{w \in C([0,T], \mathbb{R}^d) : |w(0)| \leq R,\ |w(t) - w(s)| \leq R|t-s|^\alpha\}.$$

By Ascoli theorem, K_R is compact in $C([0,T], \mathbb{R}^d)$. Let $\nu_n = \text{law}(X_n(\cdot))$. Then

$$\nu_n(K_R^c) \leq \nu_n(|w(0)| > R) + \nu_n(\exists\ t \neq s,\ |w(t) - w(s)| > R|t-s|^\alpha).$$

But for R big enough,

$$\nu_n(|w(0)| > R) = \mathbb{P}(|X_n(0)| > R) = \mu_0(|x| > R) = 0,$$

and according to (6.3),

$$\begin{aligned}\nu_n(\exists\ t \neq s,\ |w(t) - w(s)| > R|t-s|^\alpha) &= \mathbb{P}(\exists\ t \neq s,\ |X_n(t) - X_n(s)| > R|t-s|^\alpha) \\ &\leq \mathbb{P}(M_n \geq R) \leq \frac{\|M_n\|_{L^{2p}}^{2p}}{R^{2p}} \leq \frac{C_p}{R^{2p}}.\end{aligned}$$

Therefore $\sup_n \nu_n(K_R^c) < \varepsilon$ for n big enough. The family $\{\nu_n : n \geq 1\}$ is tight. Up to a subsequence, ν_n converges to ν weakly. Now let $f \in C_c^2(\mathbb{R}^d)$ and F be a bounded continuous function from $C([0,T], \mathbb{R}^d)$ to $\mathbb{R}$ which is $\mathcal{F}_s^W$-measurable. We have

$$\begin{aligned}&\mathbb{E}_\nu\bigg[\Big(f(w(t)) - f(w(s)) - \int_s^t (Lf)(w_u)du\Big)F\bigg] \\ &= \lim_{n\to\infty} \mathbb{E}\bigg[\Big(f(X_n(t)) - f(X_n(s)) - \int_s^t (Lf)(X_n(u))du\Big)F(X_n)\bigg].\end{aligned} \tag{6.4}$$

By Itô formula, we have

$$f(X_n(t)) - f(X_n(s)) = M_{s,t} + \int_s^t (Lf)(X_n(\Phi_n(u)))du,$$

where $M_{s,t}$ is the martingale part, so that

$$\mathbb{E}(M_{s,t}F(X_n)) = 0.$$

Now we will prove that

$$\lim_{n\to+\infty} \mathbb{E}\Big[\Big(\int_s^t \big((Lf)(X_n(\Phi_n(u))) - (Lf)(X_n(u))\big)du\Big)F(X_n)\Big] = 0. \tag{6.5}$$

If $X_n(\cdot)$ converges to $X(\cdot)$ uniformly over $[0,T]$, it is easy to see that (6.5) holds; this can be realized by using Skorohod representation theorem. We will give another proof using the following basic estimate, having its own interest.

Lemma 6.2 *For $1 < a < \sqrt{2}$, there exists $C > 1$ such that*

$$\mathbb{P}\Big(\sup_{0\le t\le T} |X_n(t) - X_n(\Phi_n(t))| \ge a^{-n}\Big) \le C_T e^{-C^n/4}. \tag{6.6}$$

End of the proof of Theorem 6.1. Let

$$\Omega_n = \Big\{\sup_{0\le t\le T} |X_n(t) - X_n(\Phi_n(t))| \ge a^{-n}\Big\}.$$

Let $\varepsilon > 0$, since f is compactly supported, $x \to (Lf)(x)$ is uniformly continuous on the support of f. Therefore for $n \ge 1$ big enough, for $\omega \notin \Omega_n$,

$$|(Lf)(X_n(\Phi_n(u))) - (Lf)(X_n(u))| < \varepsilon, \quad \forall\, u \in [0,T].$$

Then

$$\Big|\mathbb{E}\Big\{\Big(\int_s^t (Lf)\big(X_n(\Phi_n(u))\big) - (Lf)(X_n(u))du\Big)F(X_n)\Big\}\Big| \le \varepsilon\mathbb{P}(\Omega_n^c) + C\mathbb{P}(\Omega_n).$$

Then (6.5) follows. □

To prove Lemma 6.2, we need the following preparation.

Lemma 6.3 *Let $X_t = \int_0^t \sigma_s dB_s + \int_0^t f_s ds$ be a semi-martingale with $\sigma_s \in \mathcal{M}_{d,m}$, $f_s \in \mathbb{R}^d$. Suppose furthermore that*

$$\|\sigma_s(\omega)\| \le A, \quad |f_s(\omega)| \le B. \tag{6.7}$$

Then for $T > 0$, $R > \sqrt{d}BT$,

$$\mathbb{P}\Big(\sup_{0\le t\le T} |X_t| \ge R\Big) \le 2d\, e^{-(R-\sqrt{d}BT)^2/2dA^2T}. \tag{6.8}$$

Proof. We have $|X(t)| = \big(\sum_{i=1}^d X_i^2(t)\big)^{\frac{1}{2}} \le \sqrt{d}\max_{1\le i\le d}|X_i(t)|$, thus

$$\{|X(t)| \ge R\} \subset \bigcup_{i=1}^{d} \Big\{|X_i(t)| \ge \frac{R}{\sqrt{d}}\Big\}.$$

But

$$X_i(t) = \int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\rangle + \int_0^t \langle f_s, \varepsilon_i\rangle ds,$$

where $\{\varepsilon_1, \cdots, \varepsilon_d\}$ is the canonical basis of $\mathbb{R}^d$. Then for $t \le T$,

$$|X_i(t)| \le \Big|\int_0^t \langle \sigma_s^* \varepsilon_i, dB_t\rangle\Big| + BT.$$

Therefore

$$\Big\{\sup_{0\le t\le T} |X_i(t)| \ge \frac{R}{\sqrt{d}}\Big\} \subset \Big\{\sup_{0\le t\le T}\Big|\int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\Big| \ge \frac{R}{\sqrt{d}} - BT\Big\}. \tag{6.9}$$

Let $\alpha > 0$. We know that

$$M_t := \exp\Big\{\alpha \int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\rangle - \frac{\alpha^2}{2}\int_0^t |\sigma_s^* \varepsilon_i|^2 ds\Big\}$$

is a martingale. For $M > 0$, we have

$$\begin{aligned}\Big\{\sup_{0\le t\le T}\int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\rangle \ge M\Big\} &\subset \Big\{\sup_{0\le t\le T} M_t \ge \exp\Big(\alpha M - \frac{\alpha^2}{2}\int_0^T |\sigma_s^* \varepsilon_i|^2 ds\Big)\Big\}\\ &\subset \Big\{\sup_{0\le t\le T} M_t \ge \exp\Big(\alpha M - \frac{\alpha^2}{2}A^2 T\Big)\Big\}.\end{aligned}$$

Again by Doob's maximal inequality,

$$\mathbb{P}\Big(\sup_{0\le t\le T}\int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\rangle \ge M\Big) \le e^{-\alpha M + \frac{\alpha^2}{2}A^2 T}\mathbb{E}(M_T) = e^{-\alpha M - \frac{\alpha^2}{2}A^2 T}.$$

Taking the minimum over $\alpha > 0$, we get

$$\mathbb{P}\Big(\sup_{0\le t\le T}\int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\rangle \ge M\Big) \le e^{-M^2/2A^2T}.$$

Now

$$\begin{aligned}&\Big\{\sup_{0\le t\le T}\Big|\int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\rangle\Big| \ge M\Big\}\\ \subset &\Big\{\sup_{0\le t\le T}\int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\rangle \ge M\Big\}\bigcup\Big\{\sup_{0\le t\le T}\Big(-\int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\rangle\Big) \ge M\Big\}.\end{aligned}$$

It follows that

$$\mathbb{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \langle \sigma_s^* \varepsilon_i, dB_s\rangle\Big| \ge M\Big) \le 2e^{-M^2/2A^2T}.$$

Replacing M by $\frac{R}{\sqrt{d}} - BT$, we get the result. $\square$

Proof of Lemma 6.2. We have

$$X_n(t) - X_n(k2^{-n}) = \int_{k2^{-n}}^t \sigma\big(X_n(\Phi_n(s))\big)dB_s + \int_{k2^{-n}}^t V\big(X_n(\Phi_n(s))\big)ds.$$

Using the estimate (6.8),

$$\mathbb{P}\Big(\sup_{t\in[k2^{-n},(k+1)2^{-n}]} |X_n(t) - X_n(\Phi_n(t))| \geq a^{-n}\Big) \leq 2de^{-\frac{1}{2}\left(\frac{2}{a^2}\right)^n}.$$

Now

$$\mathbb{P}\Big(\sup_{0\leq t\leq T} |X_n(t) - X_n(\Phi_n(t))| \geq a^{-n}\Big) \leq T2^n \cdot 2de^{-\frac{1}{2}\left(\frac{2}{a^2}\right)^n} \leq 2dTe^{-\frac{1}{4}\left(\frac{2}{a^2}\right)^n}$$

for n big enough such that $n\log 2 \leq \frac{1}{4}\left(\frac{2}{a^2}\right)^n$. $\square$

Now we shall study the uniqueness in law. First, remark that if μ_B is the law of the Brownian motion $\omega \to B_\cdot(\omega)$, then

$$\begin{aligned}&\int_W \varphi(\gamma(t_1), \cdots, \gamma(t_n))d\mu_B(\gamma) \\ = &\int_{\mathbb{R}^n} \varphi(x_1, \cdots, x_n)P_{t_1}(x_1)P_{t_2-t_1}(x_2 - x_1)\cdots P_{t_n-t_{n-1}}(x_n - x_{n-1}),\end{aligned}$$

where $P_t(x) = (2\pi t)^{-d/2}e^{-|x|^2/2t}$. Therefore the law of $\omega \to B_\cdot(\omega)$ is uniquely determined.

Theorem 6.4 *Let V be a <u>bounded</u> Borel vector field on $\mathbb{R}^d$. Then the SDE*

$$dX_t = dB_t + V(X_t)dt \tag{6.10}$$

admits a weak solution and the uniqueness in law holds.

Proof. Let

$$M_T = \exp\Big\{ -\int_0^T \langle V(B_s), dB_s\rangle - \frac{1}{2}\int_0^T |V(B_s)|^2 ds\Big\}. \tag{6.11}$$

Then the Girsanov theorem says that, under the new probability measure $d\mathbb{Q} = M_T d\mathbb{P}$,

$$w_t := B_t + \int_0^t V(B_s)ds$$

is a Brownian motion. Therefore (B_t, w_t) solves (6.10) under $\mathbb{Q}$. By expression (6.11), M_T is a functional of $B:\ M_T = F(B)$. If γ is the law of SDE (6.10), then

$$\int \varphi\, d\nu = \mathbb{E}\big(\varphi(B)M_T\big) = \int \varphi\, F\, d\mu_B.$$

It follows that ν is uniquely determined. $\square$

The following general result is due to Stroock and Varadhan [11].

Theorem 6.5 *Let $a(x) = \sigma(x)\sigma(x)^*$ be a bounded continuous matrix-valued function such that $a \geq c\mathrm{Id}$, $c > 0$; V a bounded Borel vector field. Then the SDE* (6.1) *has the uniqueness in law.*

In the remainder, we shall show how the ellipticity makes a weak solution to be a strong one.

Theorem 6.6 *Under the same hypothesis as in Theorem 6.4, the SDE* (6.10) *admits a unique strong solution.*

We will not give a complete proof, but emphasize the crucial role of the ellipticity. Here are two basic results in this context.

A result from PDE: Assume $V_. \in L^{2d+2}(\mathbb{R}_+ \times \mathbb{R}^d)$. Then the backward PDE

$$\frac{\partial u_t^k}{\partial t} + \Delta u_t^k + V \cdot \nabla u_t^k = 0, \quad u_T^k(x) = x_k \in \mathbb{R} \tag{6.12}$$

admits a solution $u \in W^{1,2}_{2d+2}\big((0,T) \times B(R)\big)$; moreover there exists a small T_0 such that

$$|u_t(x) - u_t(y)| \geq c|x-y|, \quad \forall\, t \in [0,T],\, x, y \in \mathbb{R}^d. \tag{6.13}$$

Krylov estimate: Let X^0, X^1 be two solutions to (6.10) with the same (B_t) and $X^0(0) = X^1(0) = x$. For any $\alpha \in [0,1]$, set

$$X^\alpha(t) = \alpha X^1(t) + (1-\alpha) X^0(t).$$

Then for any Borel function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ and $q \geq d+1$, $\lambda > 0$, $T > 0$, we have

$$\mathbb{E}\bigg(\int_0^{\tau^R \wedge T} e^{-\lambda t} f(t, X^\alpha(t))dt\bigg) \leq N \|f\|_{L^q((0,T)\times B(R))}, \tag{6.14}$$

where $\tau^R = \inf\{t \geq 0 : |X^0(t) - x| + |X^1(t) - x| \geq R\}$. For u given in (6.12), Itô formula holds:

$$du^k(t, X^i(t)) = \frac{\partial u^k}{\partial x_j}(t, X^i(t)) dB^j(t), \quad i = 0, 1.$$

In this way, the drift term V_t disappears. Now we compute $|u(t, X^1(t)) - u(t, X^0(t))|^2$. Let $\eta_t = u(t, X^1(t)) - u(t, X^0(t))$ and $\xi_t = |\eta_t|^2$. By Itô formula,

$$\begin{aligned}
d\xi_t &= 2\langle \eta_t, d\eta_t\rangle + d\eta_t \cdot d\eta_t \\
&= \sum_{j=1}^d 2\Big\langle \eta_t, \frac{\partial u}{\partial x_j}(t, X^1(t)) - \frac{\partial u}{\partial x_j}(t, X^0(t))\Big\rangle dB^j(t) \\
&\quad + \sum_{j=1}^d \left| \frac{\partial u}{\partial x_j}(t, X^1(t)) - \frac{\partial u}{\partial x_j}(t, X^0(t))\right|^2 dt \\
&= \xi_t\,(dM_t + dN_t), \quad \xi_0 = u(0,x) - u(0,x) = 0,
\end{aligned}$$

where

$$dM_t = 2\xi_t^{-1}\sum_{j=1}^d \langle \eta_t, \frac{\partial u}{\partial x_j}(t, X^1(t)) - \frac{\partial u}{\partial x_j}(t, X^0(t))\rangle dB_t^j,$$

$$dN_t = \xi_t^{-1}\sum_{j=1}^d \left|\frac{\partial u}{\partial x_j}(t, X^1(t)) - \frac{\partial u}{\partial x_j}(t, X^0(t))\right|^2 dt.$$

In the "good situation", ξ_t satisfies a linear SDE driven by a semi-martingale with $\xi_0 = 0$. By pathwise uniqueness, $\xi_t = 0$ a.s. Then according to (6.13), $X^0(t) = X^1(t)$ for $t \le T_0$. Now for $t \in [T_0, 2T_0]$, introduce $B_t^{T_0} = B(T_0 + t) - B(T_0)$. Then $t \to X^i(t + T_0)$ satisfies

$$X^i(t + T_0) = X^i(T_0) + B_t^{T_0} + \int_0^t V(X_{s+T_0}^i)ds.$$

By what has been done in the above, we get

$$X^1(t + T_0) = X^0(t + T_0) \quad \text{for } t \le T_0.$$

Proceeding in this way, we see that $X^1(t) = X^0(t)$ for all $t \ge 0$. Now we shall show that we are in this "good situation". We have

$$dM_t \cdot dM_t = 4\xi_t^{-2}\sum_{j=1}^d \langle \eta_t, \frac{\partial u}{\partial x_j}(t, X^1(t)) - \frac{\partial u}{\partial x_j}(t, X^0(t))\rangle^2 dt. \tag{6.15}$$

By mean formula,

$$\frac{\partial u}{\partial x_j}(t, X^1(t)) - \frac{\partial u}{\partial x_j}(t, X^0(t)) = \Big(\int_0^1 \nabla_x \frac{\partial u}{\partial x_j}(t, X^\alpha(t))d\alpha\Big)(X^1(t) - X^0(t)),$$

where ∇_x denotes the gradient with respect to x, hence

$$\left|\frac{\partial u}{\partial x_j}(t, X^1(t)) - \frac{\partial u}{\partial x_j}(t, X^0(t))\right| \le |\eta_t| \int_0^1 \left|\nabla_x \frac{\partial u}{\partial x_j}(t, X^\alpha(t))\right| d\alpha.$$

Combining with (6.15), it is sufficient to show

$$\mathbb{E}\Big(\int_0^{T_0 \wedge \tau^R} \left|\nabla_x \frac{\partial u}{\partial x_j}(t, X^\alpha(t))\right|^2 dt\Big) < +\infty.$$

By Krylov estimate (6.14),

$$\mathbb{E}\Big(\int_0^{T_0\wedge\tau^R} \left|\nabla_x \tfrac{\partial u}{\partial x_j}(t, X^\alpha(t))\right|^2 dt\Big) \le e^{T_0} N \left\|\nabla_x \tfrac{\partial u}{\partial x_j}\right\|_{L^q((0,T)\times B(R))}$$
$$\le e^{T_0} N \|u\|_{W_q^{1,2}((0,T)\times B(R))}$$

which is finite due to PDE's result. Therefore

$$t \to M_{t\wedge\tau^R} + N_{t\wedge\tau^R} =: S_t$$

is a continuous L^2-semi-martingale. We have

$$d\xi_{t\wedge\tau^R} = \xi_{t\wedge\tau^R} dS_t \quad \text{and} \quad \xi_0 = 0.$$

It follows that

$$\xi_{t\wedge\tau^R} = 0.$$

Now it is not hard to prove that $\tau^R \ge T_0$, so that we get $\xi_t = 0$ a.s. for $t \in [0, T_0]$. □

7 Notes

Section 1 is essentially taken from [3] and [10].

Section 2 is taken from DiPerna and Lions [2] in which the authors treated mainly the delicate case $V \in L^1([0,T], W^{1,1}_{loc}(\mathbb{R}^d))$.

Section 3 is taken from L. Ambrosio [1], but the proof of Theorem 3.3 is slightly different.

The definitions in Section 4 follow the book of Ikeda and Watanabe [6]. The proof of Theorem 4.3 is taken from Fang and Zhang [4].

Section 5 follows the book of Kunita [8], the proof of Kolmogorov's modification theorem can be found in [6] p20.

The relations between martingale problems and second order partial differential operators are well studied in Stroock and Varadhan's book [11]. The proof of Theorem 6.1 does not use the Skorohod representation theorem; instead, we used the elementary exponential martingale estimates; the outline of the proof of Theorem 6.6 is taken from Krylov and Röckner [7].

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