Lectures on stochastic differential equations

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Abstract

In this note, we will present some new developments on stochastic differential equations.

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Let $\sigma : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$ be a continuous function, taking values in the space of (d, m)-matrices and $b : \mathbb{R}^d \to \mathbb{R}^d$ a continuous function. Consider the following Itô stochastic differential equation (abbreviated as SDE) on $\mathbb{R}^d$:

$$dX_t(\omega) = \sigma(X_t(\omega))dW_t(\omega) + b(X_t(\omega))dt, \quad X_0(\omega) = x \in \mathbb{R}^d, \tag{0.1}$$

where $t \to W_t(\omega)$ is a $\mathbb{R}^m$-valued standard Brownian motion. Since the Brownian motion is not unique (in the sense of paths), and the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ on which is defined (W_t) is not unique. How to understand (0.1)?

Definition 0.1 *(Strong Solution) If for any given Brownian motion* $(W_t)_{t\geq 0}$ *defined on a probability space* $(\Omega, \mathbb{P})$*, there exists a* $(\mathcal{F}_t^W)_{t\geq 0}$*-adapted process* $(X_t(\omega))_{0\leq t<\tau_x}$ *such that a.s.,*

$$X_t(\omega) = x + \int_0^t \sigma(X_s(\omega))dW_s + \int_0^t b(X_s(\omega))ds, \quad t < \tau_x(\omega), \tag{0.2}$$

where $\tau_x : \Omega \to [0, +\infty]$ *is the life time of* (X_t)*, which is a* $\mathcal{F}_\cdot^W$*-stopping time.*

Recall that $\mathcal{F}_t^W$ is the σ-field on Ω generated by $\{W_s(\cdot) : s \leq t\}$; τ_x is a $\mathcal{F}_\cdot^W$-stopping time in the sense that

$$\{\omega : \tau_x(\omega) \leq t\} \in \mathcal{F}_t^W, \quad \text{for all } t \geq 0.$$

If $\tau_x(\omega)$ is finite, then

$$\lim_{t\uparrow\tau_x(\omega)} |X_t(\omega)| = +\infty.$$

The uniqueness for strong solutions to (0.1) is understood as the "*pathwise uniqueness*": for two solutions $(X_t(\omega))_{0\le t<\tau_x^1}$ and $(Y_t(\omega))_{0\le t<\tau_x^2}$ to (0.1), such that $X_0 = Y_0$, then $\tau_x^1 = \tau_x^2$ and

$$\mathbb{P}(\omega:\ X_t(\omega) = Y_t(\omega) \text{ for all } t < \tau_x^1(\omega)) = 1.$$

The existence of strong solutions is very important in probabilistic modelisations.

Definition 0.2 *(Weak Solution) For the coefficients (σ, b) given, if there exists a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions on which are defined a $\mathcal{F}_t$-compatible Brownian motion $(W_t)_{t\ge 0}$ and a $\mathcal{F}_t$-adapted process $(X_t)_{0<t<\tau_x}$ such that*

$$X_t(\omega) = X_0(\omega) + \int_0^t \sigma(X_s(\omega))dW_s(\omega) + \int_0^t b(X_s(\omega))ds, \quad t < \tau_x(\omega). \tag{0.3}$$

Here are some explanations on Definition 0.2.
(i) The usual conditions mean that the sub-σ-fields $\mathcal{F}_t$ are completed by null subsets in $\mathcal{F}$ and

$$\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}.$$

(ii) $(W_t)_{t\ge 0}$ is compatible with $(\mathcal{F}_t)_{t\ge 0}$ means $\mathcal{F}_t^W \subset \mathcal{F}_t$ and for any $s < t$, $W_t - W_s$ is independent of $\mathcal{F}_s$. It is equivalent to say that $\mathcal{F}_t^W \subset \mathcal{F}_t$ and

$$\mathbb{E}\big(e^{i\langle W_t - W_s, \xi\rangle}\big|\mathcal{F}_s\big) = e^{-(t-s)|\xi|^2/2}, \quad s < t. \tag{0.4}$$

(iii) X_t is measurable with respect to $\mathcal{F}_t$, not necessarily measurable with respect to $\mathcal{F}_t^W$. If this latter situation happens, then the weak solutions become the strong ones.

The suitable notion of uniqueness for weak solutions is the "uniqueness in law". For the simplicity of exposition, we consider the case where $\tau_x = +\infty$. Then for $\omega \in \Omega$ given, $t \to X_t(\omega)$ is in $C([0, +\infty), \mathbb{R}^d)$. The law of the solution is the law of $\omega \to X_\cdot(\omega)$ on $C([0, +\infty), \mathbb{R}^d)$. The uniqueness in law means that if $\tilde{X}$ is another solution defined on another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$. Then

$$\text{law}(X_\cdot) = \text{law}(\tilde{X}_\cdot).$$

1 Strong Solutions and Euler approximations

The case where the coefficients satisfy the global Lipschitz conditions is very classical. In this section, we will consider the case where σ and b are bounded and

$$\|\sigma(x) - \sigma(y)\|^2 \le C|x-y|^2 \log\frac{1}{|x-y|}, \quad |b(x) - b(y)| \le C|x-y|\log\frac{1}{|x-y|} \tag{1.1}$$

for any $|x - y| \le \delta_0$. In this case, the Euler approximation does work.

Theorem 1.1 *Assume that the coefficients σ and b satisfy the condition* (1.1) *and are bounded:*

$$||\sigma(x)|| \le A, \quad |b(x)| \le B \text{ for all } x \in \mathbb{R}^d. \tag{1.2}$$

For $n \ge 1$, define $(X_n(t))_{n\ge 1}$ by $X_n(0) = x$ and

$$X_n(t) = X_n(k2^{-n}) + \sigma(X_n(k2^{-n}))(W_t - W_{k2^{-n}}) + b(X_n(k2^{-n}))(t - k2^{-n})$$

for $k2^{-n} \le t \le (k+1)2^{-n}$*. Then for any* $T > 0$*, almost surely,* $X_n(t)$ *converges uniformly in* $t \in [0, T]$*, to the solution* X_t *of stochastic differential equation* (0.1)*.*

Proof. Define $\phi_n(t) = k2^{-n}$ for $t \in [k2^{-n}, (k+1)2^{-n}[$, $k \ge 0$. Then $X_n(t)$ can be expressed by

$$X_n(t) = x + \int_0^t \sigma\big(X_n(\phi_n(s))\big)dW_s + \int_0^t b\big(X_n(\phi_n(s))\big)ds. \tag{1.3}$$

Let $1 < a < \sqrt{2}$. Introduce the stopping time

$$\tau_n = \inf\big\{t > 0,\ |X_n(t) - X_n(\phi_n(t))| \ge a^{-n}\big\}.$$

For $t \in [k2^{-n}, (k+1)2^{-n}[$, by expression (1.3), we have

$$\begin{aligned} X_n(t) - X_n(\phi_n(t)) = &\int_0^{t-k2^{-n}} \sigma\big(X_n(\phi_n(k2^{-n}+s))\big)d\tilde{W}_s \\ &\int_0^{t-k2^{-n}} b\big(X_n(\phi_n(k2^{-n}+s))\big)ds \end{aligned}$$

where $\tilde{W}_s = W_{k2^{-n}+s} - W_{k2^{-n}}$. Using lemma 1.6 below,

$$\begin{aligned} &P\Big(\sup_{k2^{-n} \le t < (k+1)2^{-n}} |X_n(t) - X_n(\phi_n(t))| \ge a^{-n}\Big) \\ &\le 2d \exp\big\{-(a^{-n} - \sqrt{d}B2^{-n})^2/2dA^2 2^{-n}\big\} \\ &= 2d \exp\big\{-(\frac{2}{a^2})^n(1 - \sqrt{d}B(\frac{2}{a})^{-n})^2/2dA^2\big\}. \end{aligned}$$

Let $c = 2/a^2$, which is strictly bigger than 1. Therefore for large n,

$$P\Big(\sup_{k2^{-n} \le t < (k+1)2^{-n}} |X_n(t) - X_n(\phi_n(t))| \ge a^{-n}\Big) \le 2de^{-c^n/4dA^2}$$

and for integer $T > 0$,

$$P(\tau_n \le T) \le 2d2^nT\exp\big\{-c^n/4dA^2\big\}.$$

Hence for sufficiently large n,

$$P(\tau_n \le T) \le e^{-c^n/8dA^2}. \tag{1.4}$$

Now define $\eta_n(t) = X_{n+1}(t) - X_n(t)$ and $\xi_n(t) = |\eta_n(t)|^2$. Introduce the notations

$$e_s = \sigma\big(X_{n+1}(\phi_{n+1}(s))\big) - \sigma\big(X_n(\phi_n(s))\big),$$

$$f_s = b\big(X_{n+1}(\phi_{n+1}(s))\big) - b\big(X_n(\phi_n(s))\big).$$

By lemma 1.3 below, we have

$$d\xi_n(t) = 2\big\langle e_t^*\eta_n(t), dW_t\big\rangle + 2\big\langle \eta_n(t), f_t\big\rangle dt + ||e_t||^2 dt \tag{1.5}$$

and the stochastic contraction $d\xi_n(t)\cdot d\xi_n(t)$ is given by

$$d\xi_n(t)\cdot d\xi_n(t) = 4|e_t^*\eta_n(t)|^2\, dt, \tag{1.6}$$

where e^* denotes the transpose matrix of e. Define the stopping time

$$\zeta_n = \inf\big\{t>0,\ \xi_n(t) \geq \frac{1}{n^{2\beta}}\big\} \tag{1.7}$$

where $\beta > 1$ is a parameter. Then for $s \leq \tau_{n+1}$ and n large enough, we can use (1.1) to obtain

$$\begin{aligned}
&||\sigma\big(X_{n+1}(\phi_{n+1}(s))\big) - \sigma\big(X_{n+1}(s)\big)||^2 \\
&\leq C\,\big|X_{n+1}(\phi_{n+1}(s)) - X_{n+1}(s)\big|^2 \log\Big\{1/|X_{n+1}(\phi_{n+1}(s)) - X_{n+1}(s)\big|\Big\} \\
&\leq C\, a^{-2(n+1)} \log a^{n+1} \leq C\, a^{-2n}\log a^n
\end{aligned}$$

where we used the fact that $s \to s\log 1/s$ is increasing over $[0, 1/e]$. In the same way,

$$||\sigma\big(X_n(\phi_n(s))\big) - \sigma\big(X_n(s)\big)||^2 \leq C\, a^{-2n}\log a^n,$$

and for $s \leq \tau_n \wedge \tau_{n+1} \wedge \zeta_n$,

$$\begin{aligned}
||e_s||^2 \leq 2\Big\{&||\sigma\big(X_{n+1}(\phi_{n+1}(s))\big) - \sigma\big(X_{n+1}(s)\big)||^2 \\
&+ ||\sigma\big(X_{n+1}(s)\big) - \sigma\big(X_n(s)\big)||^2 \\
&+ ||\sigma\big(X_n(\phi_n(s))\big) - \sigma\big(X_n(s)\big)||^2\Big\} \\
\leq\ & 2C\,\{\xi_n(s)\log\{1/\xi_n(s)\} + 2a^{-2n}\log a^n\}.
\end{aligned}$$

On the other hand, for $s \leq \tau_n \wedge \tau_{n+1} \wedge \zeta_n$,

$$\begin{aligned}
|\big\langle\eta_n(t), f_t\big\rangle| \leq C\,|\eta_n(t)|\Big\{&|X_{n+1}(\phi_{n+1}(t)) - X_{n+1}(t)|\,\log\{1/|X_{n+1}(\phi_{n+1}(t)) - X_{n+1}(t)|\} \\
&+ |X_n(\phi_n(t)) - X_n(t)|\,\log\{1/|X_n(\phi_n(t)) - X_n(t)|\} \\
&+ |\eta_n(t)|\,\log\{1/|\eta_n(t)|\}\Big\} \\
\leq C\,\Big\{&\xi_n(t)\log\{1/\xi_n(t)\} + \frac{2}{n^\beta}a^{-n}\log a^n\Big\}.
\end{aligned}$$

Choose the parameter ρ_n by

$$\rho_n = \frac{2}{n^\beta}a^{-n}\log a^n. \tag{1.8}$$

Then the conditions (1.20) in lemma 1.5 are satisfied with $C_3 = 2C, C_4 = C$ and δ replaced by ρ_n. Now consider the function $\psi_n(\xi) = \int_0^\xi \frac{ds}{s\log(1/s)+\rho_n}$ and $\Phi_n(\xi) = e^{4\psi_n(\xi)}$. We have

$$\Phi_n'(\xi) = \frac{4\Phi_n(\xi)}{\xi\log(1/\xi)+\rho_n} \text{ and } \Phi_n''(\xi) = \frac{4\Phi_n(\xi)(5+\log\xi)}{(\xi\log 1/\xi+\rho_n)^2} \le 0$$

for $\xi \le c_o$ small enough. The conditions in (1.19) are satisfied with $C_1 = 4, C_2 = 0$. Let

$$\tilde{\tau}_n = \tau_n \wedge \tau_{n+1} \wedge \zeta_n.$$

For large n, $\xi_n(t\wedge\tilde{\tau}_n) \le c_o$. Let $K = 24C$. Then by lemma 1.5, we have the following estimate

$$\mathbb{E}\Big(\Phi_n(\xi_n(t\wedge\tilde{\tau}_n))\Big) \le e^{Kt} \text{ for all } t,$$

from which we get

$$\mathbb{E}\Big(\mathbf{1}_{\{\tau_n\wedge\tau_{n+1}\ge T, \zeta_n\le T\}}\Phi_n(\xi_n(T\wedge\tilde{\tau}_n))\Big) \le e^{KT},$$

or

$$P(\tau_n\wedge\tau_{n+1}\ge T, \zeta_n\le T)\cdot\Phi_n(\frac{1}{n^{2\beta}}) \le e^{KT},$$

Therefore

$$P(\tau_n\wedge\tau_{n+1}\ge T, \zeta_n\le T) \le e^{KT}\exp\Big\{-4\int_0^{n^{-2\beta}}\frac{ds}{s\log 1/s+\rho_n}\Big\}. \tag{1.9}$$

Since $0 < \rho_n < n^{-2\beta}\log n^{2\beta}$, there exists $c_n \in]0, n^{-2\beta}[$ such that

$$c_n\log\frac{1}{c_n} = \rho_n = \frac{2}{n^\beta}a^{-n}\log a^n < a^{-n}\log a^n. \tag{1.10}$$

The function $s \to s\log\frac{1}{s}$ being increasing over $[0,1/e]$, from (1.10), we see that

$$0 < c_n < a^{-n}. \tag{1.11}$$

Now

$$\int_0^{n^{-2\beta}}\frac{ds}{s\log\frac{1}{s}+\rho_n} \ge \int_{c_n}^{n^{-2\beta}}\frac{ds}{2s\log\frac{1}{s}} = -\frac{1}{2}\log\Big(\frac{\log n^{-2\beta}}{\log c_n}\Big).$$

According to (1.9),

$$P(\tau_n\wedge\tau_{n+1}\ge T, \zeta_n\le T) \le e^{KT}\left(\frac{\log n^{-2\beta}}{\log c_n}\right)^2 \le e^{KT}\left(\frac{2\beta\log n}{n\log a}\right)^2,$$

where the last inequality was deduced by (1.11). Therefore for some constant $C_1 > 0$ and n big enough,

$$P(\tau_n\wedge\tau_{n+1}\ge T, \zeta_n\le T) \le C_2\Big(\frac{\log n}{n}\Big)^2. \tag{1.12}$$

Now combining (1.4) and (1.12), we get

$$P(\zeta_n \le T) \le \frac{1}{n^\gamma} \text{ for some } \gamma > 1,$$

or

$$P\Big(\sup_{0\le t\le T} |X_{n+1}(t) - X_n(t)|^2 \ge \frac{1}{n^{2\beta}}\Big) \le \frac{1}{n^\gamma}.$$

By Borel-Cantelli lemma, almost surely

$$\sup_{0\le t\le T} |X_{n+1}(t) - X_n(t)| \le \frac{1}{n^\beta} \text{ for large } n.$$

It follows that the series

$$X_t := \sum_{n\ge 1}\big(X_{n+1}(t) - X_n(t)\big) + X_1(t)$$

converges uniformly in $t \in [0, T]$. It is easy to check that X_t is the solution of stochastic differential equation (0.1). $\square$

A comment on the lifetime τ_x. By the above construction, we see that for $x \in \mathbb{R}^d$, $\tau_x = +\infty$ a.s. More precisely, let

$$A = \{(x, \omega) \in \mathbb{R}^d \times \Omega : \tau_x(\omega) = +\infty\}.$$

Let ν be a Borel probability measure on $\mathbb{R}^d$, then

$$(\nu \otimes \mathbb{P})(A) = 1.$$

By Fubini theorem,

$$\int_\Omega \Big[\int_{\mathbb{R}^d} \mathbf{1}_A(x, \omega) d\nu(x)\Big] d\mathbb{P}(\omega) = 1.$$

It follows that there exists $\Omega_0 \subset \Omega$ with full probability, such that for each $\omega \in \Omega_0$, $\tau_x(\omega) = +\infty$ for ν-a.e. x. By limit procedure, we see that $(x, \omega) \to X_t(x, \omega)$ is measurable with respect to $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t}^{\nu\otimes\mathbb{P}}$.

A new definition of strong solution: Let $W(\mathbb{R}^d) = C([0, +\infty), \mathbb{R}^d)$ and $W_0(\mathbb{R}^m) = \{w \in W(\mathbb{R}^m) : w(0) = 0\}$. Let μ_W be the law of the Brownian motion on $W_0(\mathbb{R}^m)$. We say that the SDE (0.1) has a strong solution if there exists $F : \mathbb{R}^d \times W_0(\mathbb{R}^m) \to W(\mathbb{R}^d)$ such that
(i) for any Borel probability measure ν on $\mathbb{R}^d$, F is measurable with respect to

$$\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0(\mathbb{R}^m))}^{\nu\otimes\mu_W};$$

(ii) for any t, x given, $w \to F(x, w)(t)$ is $\mathcal{F}_t^W$-measurable, where $\mathcal{F}_t^W = \sigma(w(s) : s \le t)$, and $X_t(w) = F(X_0(w), B)$ is a strong solution to (0.1) in the sense of Definition 0.1, where X_0 is random variable independent of B.

We complete this subsection by stating the following important result due to Yamada and Watanabe, whose proof can be found in any textbook on SDEs (for example [9], [23]).

Theorem 1.2 *(Yamada-Watanabe) The SDE (0.1) has a unique strong solution if and only if it admits a weak solution and the pathwise uniqueness holds.*

1.1 Appendix: preparing lemmas

We complete this section with some preliminary results, which were used in the above and will be used in the sequel. Let $(\Omega, \mathcal{F}, P)$ be a probability space, endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $(W_t)_{t\geq 0}$ be a $\mathcal{F}_t$-Brownian motion taking values in $\mathbb{R}^m$. Consider the following Itô process on $\mathbb{R}^d$:

$$\eta_t = \eta_0 + \int_0^t e_s dW_s + \int_0^t f_s ds, \quad \eta_0 \in \mathbb{R}^d \tag{1.13}$$

where $(e_t(\omega))_{t\geq 0}$ is a matrices-valued adapted process such that $\int_0^T ||e_s||^2 \, ds < +\infty$ for any $T > 0$ and $(f_t(\omega))_{t\geq 0}$ is a $\mathbb{R}^d$-valued adapted process such that $\int_0^T |f_s| \, ds < +\infty$ for any $T > 0$.

Lemma 1.3 *Let $\xi_t = |\eta_t|^2$. Then*

$$d\xi_t = 2\big\langle e_t^* \eta_t, dW_t \big\rangle + 2\big\langle \eta_t, f_t \big\rangle \, dt + ||e_t||^2 \, dt \tag{1.14}$$

where e_t^* denotes the transpose matrix of e_t. The stochastic contraction $d\xi_t \cdot d\xi_t$ is given by

$$d\xi_t \cdot d\xi_t = 4|e_t^* \eta_t|^2 \, . \tag{1.15}$$

Proof. It follows directly from Itô formula. □

Lemma 1.4 *Let ρ be a continuous function on $[0, +\infty[$ such that $\rho \geq 1$. Let Φ be a strictly positive, C^2-function on $[0, +\infty[$ satisfying the conditions*

$$|\Phi'(\xi)| \leq \frac{C_1 \Phi(\xi)}{\xi \rho(\xi) + 1}, \quad \Phi''(\xi) \leq \frac{C_2 \Phi(\xi) \rho(\xi)}{(\xi \rho(\xi) + 1)^2} \tag{1.16}$$

where C_1, C_2 are two positive constant. Keeping the notations in lemma 1.3, assume that almost surely and for all $t \geq 0$,

$$||e_t||^2 \leq C_3 \, (\xi_t \, \rho(\xi_t) + 1), \quad |f_t| \leq C_4 \, (\xi_t^{1/2} \, \rho(\xi_t) + 1) \tag{1.17}$$

where C_3, C_4 are two positive constant. Define the stopping time $\tau_R = \inf\{t > 0, \ \xi_t \geq R\}$. Let

$$K = (C_1 + 2C_2) C_3 + 4 C_1 C_4. \tag{1.18}$$

Then the following estimate holds

$$\mathbb{E}\Big(\Phi(\xi_{t\wedge\tau_R})\Big) \leq \Phi(|\eta_0|^2) \, e^{Kt}, \quad \text{for any } t \geq 0, \, R > 0.$$

Proof. Using Itô formula and according to (1.14) and (1.15), we have

$$\begin{aligned}
\Phi(\xi_{t\wedge\tau_R}) &= \Phi(\xi_0) + 2 \int_0^{t\wedge\tau_R} \Phi'(\xi_s) \big\langle e_s^* \eta_s, dW_s \big\rangle \\
&+ 2 \int_0^{t\wedge\tau_R} \Phi'(\xi_s) \big\langle \eta_s, f_s \big\rangle \, ds + \int_0^{t\wedge\tau_R} \Phi'(\xi_s) ||e_s||^2 \, ds \\
&+ 2 \int_0^{t\wedge\tau_R} \Phi''(\xi_s) \; |e_s^* \eta_s|^2 \, ds \\
&= \Phi(\xi_0) + I_1(t) + I_2(t) + I_3(t) + I_4(t)
\end{aligned}$$

respectively. By conditions (1.17), we see that $I_1(t)$ is a martingale; therefore $\mathbb{E}(I_1(t)) = 0$. Using (1.16) and (1.17),

$$|\Phi'(\xi_s)\,\langle \eta_s, f_s\rangle| \le \frac{C_1C_4\Phi(\xi_s)}{\xi_s\rho(\xi_s)+1}\cdot|\eta_s|\;(|\eta_s|\rho(\xi_s)+1)$$
$$= C_1C_4\Phi(\xi_s)\frac{\xi_s\rho(\xi_s)+\xi_s^{1/2}}{\xi_s\rho(\xi_s)+1}.$$

Since $\rho \ge 1$ and

$$\sup_{\xi\ge 0}\frac{\xi\rho(\xi)+\xi^{1/2}}{\xi\rho(\xi)+1} \le 2,$$

we get the estimate

$$\mathbb{E}(I_2(t)) \le 4C_1C_4\int_0^t \mathbb{E}(\Phi(\xi_{s\wedge\tau_R}))\,ds.$$

In the same way, $\mathbb{E}(I_3(t)) \le C_1C_3\int_0^t \mathbb{E}(\Phi(\xi_{s\wedge\tau_R}))\,ds$. Now

$$\Phi''(\xi_s) \le \frac{C_2\Phi(\xi_s)\rho(\xi_s)}{(\xi_s\rho(\xi_s)+1)^2} \le \frac{C_2\Phi(\xi_s)}{\xi_s(\xi_s\rho(\xi_s)+1)}$$

and

$$\Phi''(\xi_s)|e_s^*\eta_s|^2 \le \frac{C_2C_3\Phi(\xi_s)}{\xi_s(\xi_s\rho(\xi_s)+1)}\cdot\xi_s(\xi_s\rho(\xi_s)+1) = C_2C_3\Phi(\xi_s),$$

so that $E(I_4(t)) \le 2C_2C_3\int_0^t \mathbb{E}(\Phi(\xi_{s\wedge\tau_R}))\,ds$. Let K be the constant defined in (1.18), we obtain the inequality

$$\mathbb{E}(\Phi(\xi_{t\wedge\tau_R})) \le \Phi(\xi_0) + K\int_0^t \mathbb{E}(\Phi(\xi_{s\wedge\tau_R}))\,ds.$$

By Gronwall lemma, we get that for all $t \ge 0$ and $R > 0$,

$$\mathbb{E}(\Phi(\xi_{t\wedge\tau_R})) \le \Phi(\xi_0)e^{Kt}.$$

Lemma 1.5 *Let r be a continuous function defined on a neighborhood of 0, say $]0, c_o]$, such that $r \ge 1$. Let Φ be a strictly positive, C^2-function defined on $[0, c_o]$. Suppose that there exists $\delta > 0$ such that*

$$|\Phi'(\xi)| \le \frac{C_1\Phi(\xi)}{\xi r(\xi)+\delta}, \quad \Phi''(\xi) \le \frac{C_2\Phi(\xi)r(\xi)}{(\xi r(\xi)+\delta)^2}. \tag{1.19}$$

Keep the notations in lemma 1.3 and suppose that $|\eta_0|^2 < c_o$. Define the stopping time

$$\tau = \inf\{t > 0,\ \xi_t \ge c_o\}.$$

Assume that for $t < \tau$,

$$||e_t||^2 \leq C_3\,(\xi_t r(\xi_t) + \delta), \quad |\langle \eta_t, f_t \rangle| \leq C_4\,(\xi r(\xi_t) + \delta). \tag{1.20}$$

Let

$$K = (C_1 + 2C_2)C_3 + 4C_1C_4. \tag{1.21}$$

Then

$$\mathbb{E}\Big(\Phi(\xi_{t\wedge\tau})\Big) \leq \Phi(|\eta_0|^2)\,e^{Kt}, \quad \text{for any } t \geq 0.$$

Proof. Using Itô formula and according to (1.14) and (1.15), we have

$$\begin{aligned}
\Phi(\xi_{t\wedge\tau}) &= \Phi(\xi_0) + 2\int_0^{t\wedge\tau} \Phi'(\xi_s)\langle e_s^*\eta_s, dW_s\rangle \\
&+ 2\int_0^{t\wedge\tau} \Phi'(\xi_s)\langle \eta_s, f_s\rangle\,ds + \int_0^{t\wedge\tau} \Phi'(\xi_s)||e_s||^2\,ds \\
&+ 2\int_0^{t\wedge\tau_R} \Phi''(\xi_s)|e_s^*\eta_s|^2\,ds \\
&= \Phi(\xi_0) + I_1(t) + I_2(t) + I_3(t) + I_4(t)
\end{aligned}$$

respectively. By assumption (1.20), for any $s \leq \tau$,

$$|e_s^*\eta_s|^2 \leq ||e_s||^2|\eta_s|^2 \leq C_3\,(\xi_s r(\xi_s) + \delta)\xi_s.$$

According to (1.19),

$$|\Phi'(\xi_s)e_s^*\eta_s|^2 \leq C_1^2C_3\Phi(\xi_s)^2\frac{\xi_s\,(\xi_s r(\xi_s) + \delta)}{(\xi_s r(\xi_s) + \delta)^2} \leq C_1^2C_3 \sup_{0\leq\xi\leq c_o} \Phi(\xi)^2 < +\infty.$$

Therefore $I_1(t)$ is a martingale and $\mathbb{E}(I_1(t)) = 0$. On the other hand, by assumption (1.19) and (1.20),

$$|\Phi'(\xi_s)\langle \eta_s, f_s\rangle| \leq C_1C_4\Phi(\xi_s) \text{ and } |\Phi'(\xi_s)|\,||e_s||^2 \leq C_1C_3\Phi(\xi_s)$$

and

$$\Phi''(\xi_s)\,|e_s^*\eta_s|^2 \leq C_2C_3\Phi(\xi_s).$$

Let K be the constant defined in (1.21). We have

$$\mathbb{E}\Big(\Phi(\xi_{t\wedge\tau})\Big) \leq \Phi(|\eta_0|^2) + K\int_0^t \mathbb{E}\Big(\Phi(\xi_{s\wedge\tau})\Big)\,ds.$$

It follows that $\mathbb{E}\Big(\Phi(\xi_{t\wedge\tau})\Big) \leq \Phi(|\eta_0|^2)\,e^{K\,t}$ for all $t > 0$. □

Lemma 1.6 *Keeping the same notations, assume that the coefficients e and f are bounded, namely*

$$||e_t(w)|| \leq A, \quad |f_t(w)| \leq B \quad \text{for all } (t, \omega).$$

Then for any $T > 0$ and $R > \sqrt{d}BT$, we have

$$P\Big(\sup_{0\leq s\leq T} |\xi_s| \geq R\Big) \leq 2de^{-(R-\sqrt{d}BT)^2/2dA^2T}. \tag{1.22}$$

Proof. It is a classical result and can be deduced from exponential martingale method. For a proof, see for example [22], p.81. □

2 Pathwise uniqueness, non-contact property and flows

We will study some qualitative properties to the SDE (0.1) under non-Lipschitz conditions. We begin with

Theorem 2.1 *Let r be a strictly positive, C^1-function defined on a neighborhood $]0, c_o]$ of 0, satisfying (i) $\lim_{s\to 0} r(s) = +\infty$, (ii) $\lim_{s\to 0} \frac{sr'(s)}{r(s)} = 0$ and (iii) $\int_0^a \frac{ds}{sr(s)} = +\infty$ for any $a > 0$. Assume that for $|x-y| \leq c_o$,*

$$||\sigma(x) - \sigma(y)||^2 \leq C\,|x-y|^2\, r(|x-y|^2), \quad |b(x) - b(y)| \leq C\,|x-y|\, r(|x-y|^2). \tag{2.1}$$

Then the pathwise uniqueness holds for SDE (0.1).

Proof. Without loss of generality, we can assume the lifetime ζ of stochastic differential equation (0.1) is infinite; otherwise we have the pathwise uniqueness up to the lifetime. Let X_t and Y_t be two solutions of (0.1) having the same initial data. Consider $\eta_t = X_t - Y_t$ and $\xi_t = |\eta_t|^2$. According to notations in lemma 1.3,

$$e_t = \sigma(X_t) - \sigma(Y_t), \quad f_t = b(X_t) - b(Y_t).$$

Let $\tau = \inf\{t > 0,\ \xi_t \geq c_o^2\}$. By hypothesis (2.1), for $t \leq \tau$,

$$||e_t||^2 \leq C\,\xi_t r(\xi_t) \quad \text{and} \quad |\langle \eta_t, f_t\rangle| \leq C\,\xi_t r(\xi_t).$$

According to the condition (i) on the function r, we can assume that $r(\xi) \geq 1$ for all $\xi \in]0, c_o]$. Let $\delta > 0$, we define

$$\psi_\delta(\xi) = \int_0^\xi \frac{ds}{sr(s) + \delta} \quad \text{and} \quad \Phi_\delta(\xi) = e^{\psi_\delta(\xi)}.$$

By condition (iii) on r, we see that $\Phi_0(\xi) = +\infty$ for any $\xi > 0$ and $\Phi_0(0) = 1$. We have

$$\Phi_\delta'(\xi) = \frac{\Phi_\delta(\xi)}{\xi\, r(\xi) + \delta}, \quad \Phi_\delta''(\xi) = \Phi_\delta(\xi) \frac{1 - r(\xi) - \xi\, r'(\xi)}{(\xi\, r(\xi) + \delta)^2}.$$

By conditions (i) and (ii) on the function r, there exists a large constant $C_1 > 0$ such that

$$|1 - r(\xi) - \xi\, r'(\xi)| \leq C_1\, r(\xi)$$

so that

$$\Phi_\delta''(\xi) \leq C_1 \frac{\Phi_\delta(\xi) r(\xi)}{(\xi\, r(\xi) + \delta)^2}.$$

The conditions in (1.19) are satisfied. Now using lemma 1.5, there exists a constant $C_2 > 0$ such that for any $t > 0$,

$$\mathbb{E}\Big(\Phi_\delta(\xi_{t\wedge\tau})\Big) \leq e^{C_2\, t}.$$

Letting $\delta \downarrow 0$ in the above inequality, we get

$$\mathbb{E}\Big(e^{\psi_0(\xi_{t\wedge\tau})}\Big) \leq e^{C_2\, t}$$

which implies that for t given,

$$\xi_{t\wedge\tau} = 0 \quad \text{almost surely.} \tag{2.2}$$

If $P(\tau < +\infty) > 0$, then for some large $T > 0$, $P(\tau \le T) > 0$. By (2.2), almost surely for all $t \in \mathbf{Q} \cap [0,T]$, $\xi_{t\wedge\tau} = 0$. It follows that on $\{\tau \le T\}$, $\xi_\tau = 0$ which is absurd by the definition of τ. Therefore $\tau = +\infty$ almost surely. So for any given t , $\xi_t = 0$ almost surely. Now by the continuity of samples, the two solutions are indistinguishable. □

It is known that for ordinary differential equations, pathwise uniqueness is equivalent to the property of non confluence by reversing the time. For stochastic differential equations, reversing the time is delicate. However, using this idea, we shall construct another function Φ_δ so that we shall obtain the following result

Theorem 2.2 *Under the same hypothesis as in theorem 2.1 and assume that the solution does not explode at a finite time. Then for $x_o \neq y_o$, almost surely $X_t(x_o) \neq X_t(y_o)$ for all $t > 0$.*

Such kind of non confluence property was studied by M. Emery in an early work [4] for general stochastic differential equations under Lipschitz conditions, and by T. Yamada and Y. Ogura for non-Lipschitz case in [24]. However the mixing condition imposed in [24] for coefficients σ and b seems difficult to check and not natural.

Proof. Without loss of generality, we may assume that $|x_o - y_o| < c_o/2$. Let $0 < \varepsilon < |x_o - y_o|$ and define

$$\hat\tau_\varepsilon = \inf\{t > 0,\ |X_t(x_o) - X_t(y_o)| \le \varepsilon\}, \quad \hat\tau = \inf\{t > 0,\ X_t(x_o) = X_t(y_o)\}.$$

It is clear that $\hat\tau_\varepsilon \uparrow \hat\tau$ as $\varepsilon \downarrow 0$. Let

$$\tau = \inf\{t > 0,\ |X_t(x_o) - X_t(y_o)| \ge \frac{3}{4} c_o\}.$$

Consider $\eta_t = X_{t\wedge\hat\tau_\varepsilon}(x_o) - X_{t\wedge\hat\tau_\varepsilon}(y_o)$ and $\xi_t = |\eta_t|^2$. Then using notations in lemma 1.3, we have expressions

$$e_t = \mathbf{1}_{\{\hat\tau_\varepsilon \ge t\}}\big(\sigma(X_t(x_o)) - \sigma(X_t(y_o))\big), \quad f_t = \mathbf{1}_{\{\hat\tau_\varepsilon \ge t\}}\big(b(X_t(x_o)) - b(X_t(y_o))\big).$$

By hypothesis of (2.1), for $t < \tau$,

$$||e_t||^2 \le C\, \xi_t\, r(\xi_t) \quad \text{and} \quad |\langle \eta_t, f_t\rangle| \le C\, \xi_t^{1/2}\, r(\xi_t).$$

Now define the functions

$$\psi_\delta(\xi) = \int_\xi^{c_o} \frac{ds}{sr(s) + \delta} \text{ and } \Phi_\delta = e^{\psi_\delta(\xi)} \text{ for } \xi \le c_o.$$

We have $|\Phi_\delta'(\xi)| = \dfrac{\Phi_\delta(\xi)}{\xi\, r(\xi) + \delta}$ and for some large constant $C_1 > 0$,

$$\Phi_\delta''(\xi) = \Phi_\delta(\xi) \frac{1 + r(\xi) + \xi\, r'(\xi)}{(\xi\, r(\xi) + \delta)^2} \le C_1\, \frac{\Phi_\delta(\xi) r(\xi)}{(\xi\, r(\xi) + \delta)^2}.$$

So we can apply lemma 1.5 to get the inequality

$$\mathbb{E}\Big(\Phi_\delta(\xi_{t\wedge\tau})\Big) \le \Phi_\delta(\xi_o)e^{C_2\,t} \text{ for some } C_2 > 0 \text{ and for all } t > 0.$$

Letting $\delta \downarrow 0$, by Fatou lemma, we get

$$\mathbb{E}\Big(\Phi_0(\xi_{t\wedge\tau})\Big) \le \Phi_0(\xi_o)e^{C_2\,t}.$$

Replacing ξ_t by its expression, we have

$$\mathbb{E}\Big(\Phi_0(|X_{t\wedge\hat\tau_\varepsilon\wedge\tau}(x_o) - X_{t\wedge\hat\tau_\varepsilon\wedge\tau}(y_o)|^2)\Big) \le \Phi_0(\xi_o)e^{C_2\,t}.$$

On subset $\{\hat\tau_\varepsilon < t\wedge\tau\}$, $|X_{t\wedge\hat\tau_\varepsilon\wedge\tau}(x_o) - X_{t\wedge\hat\tau_\varepsilon\wedge\tau}(y_o)| = \varepsilon$. From the above inequality, we obtain

$$P(\hat\tau_\varepsilon < t\wedge\tau)\Phi_0(\varepsilon^2) \le \Phi_0(\xi_o)e^{C_2\,t},$$

or

$$P(\hat\tau_\varepsilon < t\wedge\tau) \le \exp\Big\{-\int_{\varepsilon^2}^{\xi_o}\frac{ds}{sr(s)}\Big\} \to 0 \text{ as } \varepsilon\downarrow 0.$$

Therefore $P(\hat\tau_\varepsilon < t\wedge\tau) = 0$ for all t. Letting $t\to\infty$ we get $P(\hat\tau < \tau) = 0$. Therefore, $\xi_.$ is positive almost surely on the interval $[0,\tau]$. Now define $T_0 := 0$,

$$T_1 := \tau, \qquad T_2 = \inf\big\{t > 0,\ |X_t(x_o) - X_t(y_o)| \le \frac{c_o}{2}\big\}$$

and generally

$$T_{2n} = \inf\big\{t > T_{2n-1},\ |X_t(x_o)-X_t(y_o)| \le \frac{c_o}{2}\big\}, \quad T_{2n+1} = \inf\big\{t > T_{2n},\ |X_t(x_o)-X_t(y_o)| \ge \frac{3c_o}{4}\big\}$$

Clearly $T_n\to\infty$ almost surely as $n\to\infty$. By definition, $\xi_.$ is positive on the interval $[T_{2n-1}, T_{2n}]$. By pathwise uniqueness of solutions, X enjoys the strong Markovian property. Starting again from T_{2n} and applying the same arguments as in the first part of the proof, one can show that $\xi_.$ is positive almost surely also on the interval $[T_{2n}, T_{2n+1}]$. This completes the proof. □

In what follows, we will study the continuous modification of $X_t(x_o)$. We start with the following lemmas.

Lemma 2.3 *Let r be a strictly positive continuous function defined on $]0,c_o]$, where $0 < c_o < 1$. Assume that the coefficients σ and b are compactly supported, say,*

$$\sigma(x) = 0 \text{ and } b(x) = 0 \text{ for } |x| \ge R, \tag{2.3}$$

and satisfy the hypothesis (2.1) *in theorem* 2.1. *Let $p\ge 1$. If $s\to r(s)$ is decreasing on $]0,c_o]$, then there exists a constant $C_p > 0$ such that for all $|x|\le R+1, |y|\le R+1$,*

$$\begin{aligned} ||\sigma(x)-\sigma(y)||^2 &\le C_p\,|x-y|^2\,r\Big(\frac{|x-y|^{2p}}{M^p}\Big),\\ |b(x)-b(y)| &\le C_p\,|x-y|\,r\Big(\frac{|x-y|^{2p}}{M^p}\Big)\end{aligned} \tag{2.4}$$

where $M = \dfrac{4(R+1)^2}{c_o}$.

Proof. We only prove the conclusion for b. If $|x-y| \le c_o$, by hypothesis $(H2)$,

$$|b(x)-b(y)| \le C\ |x-y| r(|x-y|^2) \le C\ |x-y|\, r\Big((\frac{|x-y|^2}{M})^p \Big), \tag{2.5}$$

as r is supposed to be decreasing. Remark that

$$\inf_{c_o \le \xi \le 2(R+1)} \xi\, r\Big(\Big[\frac{\xi^2}{M}\Big]^p \Big) \ge c_o\, r(c_o^p) > 0$$

and $\sup_{x,y} |b(x)-b(y)| \le 2||b||_\infty$, where $||b||_\infty$ denotes the uniform norm of b over $\mathbb{R}^d$. Therefore there exists a constant $C_p > 0$ such that

$$|b(x)-b(y)| \le C_p\, |x-y|\, r\Big((\frac{|x-y|^2}{M})^p \Big) \quad \text{for} \quad |x-y| \ge c_o. \tag{2.6}$$

Combining (2.5) and (2.6), we get the result. □

Lemma 2.4 *Let σ and b be continuous functions satisfying the support condition* (2.3). *If the stochastic differential equation (0.1) has the pathwise uniqueness, then for any $|x_o| \le R+1$, $|X_t(x_o)| \le R+1$ almost surely for all $t > 0$.*

Proof. Define the stopping time

$$\tau = \inf\big\{ t \ge 0;\ |X_t(x_o)| \ge R+1 \big\}$$

Set $Y_t = X_{t\wedge\tau}(x)$. Then

$$Y_t = x_o + \int_0^{t\wedge\tau} \sigma(Y_s) dW_s + \int_0^{t\wedge\tau} b(Y_s) ds.$$

We have

$$\begin{aligned} \mathbb{E}\Big(\int_0^t ||\sigma(Y_s)||^2 \big(\mathbf{1}_{(s<\tau)} - 1 \big)^2\, ds \Big) &= \mathbb{E}\Big(\int_\tau^t ||\sigma(Y_s)||^2\, ds \Big) \\ &= \mathbb{E}\Big(\int_\tau^t ||\sigma(X_\tau)||^2\, ds \Big) \end{aligned}$$

X_τ being on the sphere of radius $R+1$, $\sigma(X_\tau) = 0$ by hypothesis (2.3), the last term in the above equality is equal to zero. Therefore $t \ge 0$,

$$\int_0^{t\wedge\tau} \sigma(Y_s) dW_s = \int_0^t \sigma(Y_s) dW_s \quad \text{and} \quad \int_0^{t\wedge\tau} b(Y_s) ds = \int_0^t b(Y_s) ds,$$

almost surely. We see that $\{Y_t,\ t \ge 0\}$ satisfies the same stochastic differential equation as $\{X_t,\ t \ge 0\}$. By pathwise uniqueness, we conclude that $Y_t = X_t$ almost surely for all $t \ge 0$, which proves the lemma. □

Lemma 2.5 *Assume the same hypothesis as in lemma 2.4 and furthermore r satisfies the condition (i)-(iii) in theorem 2.1 and* $\xi \to \xi r(\xi)$ *is concave over* $]0, c_o]$. *Let* $p \geq 2$ *be an integer. For* $|x_o| \leq R+1$ *and* $|y_o| \leq R+1$, *set*

$$\eta_t = X_t(x_o) - X_t(y_o),\ \xi_t = |\eta_t|^2 \quad \text{and} \quad z_t = (\frac{\xi_t}{M})^p$$

where M *is the constant defined in Lemma 2.3. Put* $\varphi(t) = \mathbb{E}(z_t)$. *Then for some constant* C_p,

$$\varphi'(t) \leq C_p\, \varphi(t)\, r(\varphi(t)). \tag{2.7}$$

Proof. By hypothesis imposed on the function r, we can apply lemma 2.3 and 2.4 so that z_t is a bounded process. Let $e_t = \sigma(X_t(x_o)) - \sigma(X_t(y_o))$ and $f_t = b(X_t(x_o)) - b(X_t(y_o))$. By Itô formula and applying (1.14) and (1.15),

$$\begin{aligned} dz_t = \frac{1}{M^p}\Big(& 2p\xi^{p-1}\langle e_t^*\eta_t, dW_t\rangle + 2p\xi^{p-1}\langle \eta_t, f_t\rangle\, dt \\ &+ p\xi_t^{p-1}\, ||e_t||^2\, dt + 2p(p-1)\xi_t^{p-2}|e_t^*\eta_t|^2\, dt\Big). \end{aligned}$$

By lemma 2.4, $X_t(x_o)$ and $X_t(y_o)$ are bounded by $R+1$. Now using (2.4), we have

$$\frac{\xi_s^{p-1}}{M^p}\big|\langle \eta_s, f_s\rangle\big| \leq C_p\, \frac{\xi_s^p}{M^p} r\big(\frac{\xi_s^p}{M^p}\big) = C_p z_s r(z_s) \tag{2.8}$$

and

$$\xi_s^{p-1}||e_s||^2 \leq C_p z_s r(z_s), \quad \frac{\xi_s^{p-2}}{M^p}\big|e_s^*\eta_s\big|^2 \leq C_p z_s r(z_s). \tag{2.9}$$

By concavity of $\xi \to \xi r(\xi)$ over $]0, c_o]$, we see that $\sup_{0<\xi\leq c_o} \big(\xi r(\xi)\big)$ is finite. Therefore the first term in the expression of dz_t is a martingale and $\varphi(t) = \mathbb{E}(z_t)$ is a derivable function with respect to t and

$$\begin{aligned} \varphi'(t) =& \frac{1}{M^p}\Big(2p\mathbb{E}(\xi_t^{p-1}\langle \eta_t, f_t\rangle) + p\mathbb{E}(\xi_t^{p-1}\, ||e_t||^2) \\ &+ 2p(p-1)\mathbb{E}(\xi_t^{p-2}|e_t^*\eta_t|^2)\Big) \end{aligned}$$

which is less, according to (2.8) and (2.9), than

$$(2p^2 + p)C_p\, \mathbb{E}(z_t r(z_t)) \leq (2p^2 + p)C_p\, \varphi(t) r(\varphi(t)).$$

So we get the result. □

Assume now that for $|x - y| \leq c_o$, where c_0 is a small enough constant,

$$||\sigma(x) - \sigma(y)||^2 \leq C\,|x-y|^2 \log\frac{1}{|x-y|}, \quad |b(x) - b(y)| \leq C\,|x-y| \log\frac{1}{|x-y|}. \tag{2.10}$$

Theorem 2.6 *Assume that the hypothesis* (2.10) *holds and* $P(\tau_x = +\infty,\ \text{for all } x \in \mathbb{R}^d) = 1$ *where* τ_x *is the lifetime for SDE* (0.1). *Then there exists a version* $\tilde{X}_t(x_o)$ *of* $X_t(x_o)$ *such that* $(t, x_o) \to \tilde{X}_t(x_o)$ *is continuous over* $[0, +\infty[\times\mathbb{R}^d$ *almost surely.*

Proof. We split the proof into two steps.

Step 1. Assume that σ and b are compactly supported, say,

$$\sigma(x) = 0 \quad \text{and} \quad b(x) = 0 \quad \text{for } |x| \geq R.$$

Let φ be defined as in Lemma 2.5. Solving (2.7) with $r(s) = \log\frac{1}{s}$, we get $\varphi(t) \leq (\varphi(0))^{e^{-C_p t}}$ or explicitly

$$\mathbb{E}\Big(|X_t(x_o) - X_t(y_o)|^{2p}\Big) \leq C_p\, |x_o - y_o|^{2pe^{-C_p t}}.$$

On the other hand, it is easy to see that

$$\mathbb{E}\Big(|X_t(x_o) - X_s(x_o)|^{2p}\Big) \leq C_p |t-s|^p.$$

Therefore,

$$\mathbb{E}\Big(|X_t(x_o) - X_s(y_o)|^{2p}\Big) \leq C_p\,\Big[|t-s|^p + |x_o - y_o|^{2pe^{-C_p t}}\Big]. \tag{2.11}$$

Fix $p > d+1$. Choose a constant $T_o > 0$ small enough such that $2pe^{-C_p T_0} > d+1$. It follows from (2.11) and Kolmogorov's modification theorem that there exists a version of $X_t(x_o, w)$, denoted by $\tilde{X}_t(x_o, w)$, such that $(t, x_o) \to \tilde{X}_t(x_o, w)$ is continuous over $[0, T_o] \times \{|x_o| \leq R+1\}$ almost surely. But

$$X_t(x_o, w) = x_o \quad \text{if} \quad |x_o| > R.$$

We conclude that $(t, x_o) \to \tilde{X}_t(x_o, w)$ can be extended continuously to $[0, T_o] \times \mathbb{R}^d$. Let $(\theta_{T_o} w)(t) = w(t + T_o) - w(T_o)$. Define for $0 < t \leq T_o$,

$$\tilde{X}_{T_o+t}(x_o, w) = \tilde{X}_t\big(\tilde{X}_{T_o}(x_o, w), \theta_{T_o} w\big).$$

Then $\tilde{X}_{T_o+\cdot}(x_o, w)$ satisfies the stochastic differential equation (0.1) driven by the Brownian motion $\theta_{T_o} w$ with the initial condition $\tilde{X}_{T_o}(x_o, w)$. By pathwise uniqueness, we see that $\tilde{X}_{T_o+t}(x_o, w) = X_{T_o+t}(x_o, w)$ almost surely for all $t \in [0, T_o]$. This means that $\tilde{X}_t(x_o, w)$ is a continuous version of $X_t(x_o, w)$ over $|0, 2T_o] \times \mathbb{R}^d$. Continuing in this way, we get a continuous version on the whole space $[0, +\infty[\times\mathbb{R}^d$.

Step 2: General case. We shall proceed as in [19] for locally Lipschitzian coefficients. For $R > 0$, let $f_R(x)$ denote a smooth function with compact support satisfying

$$f_R(x) = 1 \quad \text{for } |x| \leq R \quad \text{and} \quad f_R(x) = 0 \quad \text{for } |x| > R+1.$$

Define

$$\sigma_R(x) = \sigma(x) f_R(x) \quad \text{and} \quad b_R(x) = b(x) f_R(x).$$

Let $X_t^R(x, w)$ be the unique solution to the SDE (0.1) with σ and b replaced by σ_R and b_R. Let $\tilde{X}_t^R(x, w)$ denote a continuous version of $X_t^R(x, w)$. Such a version exists according to step 1. For $K > 0$, set

$$\tau_K^R(x) = \inf\{t > 0;\ |\tilde{X}_t^R(x,w)| \geq K\},$$
$$\tau_K(x) = \inf\{t > 0;\ |X_t(x,w)| \geq K\}.$$

By pathwise uniqueness, for $|x| \leq R$,

$$X_t(x,w) = \tilde{X}_t^N(x,w) \text{ for any } N > R+1 \text{ and } t < \tau_{R+1}^N,$$

or

$$\tau_{R+1}(x) = \tau_{R+1}^N(x) \text{ for any } N > R+1.$$

For $|x| \leq R$, we define

$$\tilde{X}_t(x,w) = \tilde{X}_t^{R+2}(x,w) \quad \text{on} \quad [0, \tau_{R+1}^{R+2}(x)[.$$

Then $\tilde{X}_\cdot(x,w)$ is a version of $X_\cdot(x,w)$. Let us prove that $\tilde{X}_t(x,w)$ is continuous in (t,x) for almost all w. Fix x_0 with $|x_0| \leq R$. Since the lifetime of the solution is infinite, there exists $R > 0$ such that $\tau_{R+1}^{R+2}(x_0) > t+\varepsilon$ for a small ε. This implies that $\sup_{0\leq s\leq t+\varepsilon} |\tilde{X}_s^{R+2}(x_o,w)| < R+1$. By the continuity, we can find a neighborhood $B_\delta(x_o)$ of x_o such that $\sup_{0\leq s\leq t+\varepsilon} |\tilde{X}_s^{R+2}(x,w)| < R+1$ or $\tau_{R+1}^{R+2}(x) > t+\varepsilon$ for all $x \in B_\delta(x_o)$. Hence, $\tilde{X}_s(x,w) = \tilde{X}_s^{R+2}(x,w)$ for all $x \in B_\delta(x_o)$ and $s \leq t+\varepsilon$, which implies that $\tilde{X}_s(x_o,w)$ is continuous at the point (t,x_o). □

In order to establish the property of the flow of homeomorphisms, we suppose the following stronger condition than (2.10):

$$\begin{aligned} &||\sigma(x) - \sigma(y)||^2 \leq C\,|x-y|^2 \log\left(|x-y|^{-1} + e\right), \\ &|b(x) - b(y)| \leq C\,|x-y| \log\left(|x-y|^{-1} + e\right). \end{aligned} \tag{2.12}$$

Notice that the above conditions say that for $|x-y|$ large, the coefficients σ and b behave as Lipschitzian functions. Following the approach of H. Kunita, the key step to prove the injectivity is the following negative moment estimate, which is due to Xicheng Zhang [20]:

Proposition 2.7 *Let $p \geq 2$. There exist a small time $T_0 > 0$ and a constant $C > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t \in [0, T_0]$,*

$$\mathbb{E}\Big(|X_t(x) - X_t(y)|^{-2p}\Big) \leq C\,(|x-y|^{-4p} + 1). \tag{2.13}$$

Proof. Let $\eta_t = X_t(x) - X_t(y)$ and $\xi_t = |\eta_t|^2$. Set $z_t = \xi_t^{-p}$. Assume that $|x-y| \geq \varepsilon > 0$. Let $\tau_\varepsilon = \inf\{t > 0,\ \xi_t < \varepsilon^2\}$. By Itô formula,

$$z_{t\wedge\tau_\varepsilon} = z_0 + \int_0^{t\wedge\tau_\varepsilon} z_s\,(dM_s + f_s ds), \tag{2.14}$$

where

$$dM_t = -2p\xi_t^{-1}\,\langle \big(\sigma(X_t(x)) - \sigma(X_t(y))^*\eta_t,\ dW_t\rangle, \tag{2.15}$$

and

$$\begin{aligned} f_t = &-2p\xi_t^{-1}\langle \eta_t, b(X_t(x)) - b(X_t(y))\rangle - p\xi_t^{-1}||\sigma(X_t(x)) - \sigma(X_t(y))||^2 \\ &+ 2p(p+1)\xi_t^{-2}|\big(\sigma(X_t(x)) - \sigma(X_t(y))^*\eta_t|^2. \end{aligned} \tag{2.16}$$

Notice that $t \to M_t + \int_0^t f_s\,ds$ is a semi-martingale up to the stopping time τ_ε. So we can write down z_t in the form

$$z_{t\wedge\tau_\varepsilon} = z_0\,\exp\Big\{M_{t\wedge\tau_\varepsilon} - \frac{1}{2} < M >_{t\wedge\tau_\varepsilon} + \int_0^{t\wedge\tau_\varepsilon} f_s\,ds\Big\},$$

where, according to conditions (2.12), (2.15) and (2.16),

$$< M >_{t\wedge\tau_\varepsilon} + 2\int_0^{t\wedge\tau_\varepsilon} f_s\,ds \le 2p(p+3)C\int_0^{t\wedge\tau_\varepsilon} \log\left(|\eta_t|^{-1}+e\right)ds. \tag{2.17}$$

By Cauchy-Schwarz inequality

$$\mathbb{E}(z_{t\wedge\tau_\varepsilon}) \le z_0\Big(\mathbb{E}\Big[e^{2M_{t\wedge\tau_\varepsilon}-2<M>_{t\wedge\tau_\varepsilon}}\Big]\Big)^{1/2}\times\Big(\mathbb{E}\Big[e^{<M>_{t\wedge\tau_\varepsilon}+2\int_0^{t\wedge\tau_\varepsilon} f_s\,ds}\Big]\Big)^{1/2}.$$

Let $T_0 = \frac{1}{(p+3)C}$. Then for $t\in[0,T_0]$ and according to (2.17)

$$\begin{aligned}
\mathbb{E}\Big[e^{<M>_{t\wedge\tau_\varepsilon}+2\int_0^{t\wedge\tau_\varepsilon} f_s\,ds}\Big] &\le \mathbb{E}\Big[\exp\Big\{\frac{1}{T_0}\int_0^{T_0}\log\left(|\eta_t|^{-1}+e\right)(2p)\mathbf{1}_{s\le t\wedge\tau_\varepsilon}\,ds\Big]\\
&\le \frac{1}{T_0}\int_0^{T_0}\mathbb{E}\Big[(|\eta_t|^{-1}+e)^{2p\mathbf{1}_{s\le t\wedge\tau_\varepsilon}}\Big]\,ds\\
&\le \frac{1}{T_0}\int_0^{t}\mathbb{E}\big[(|\eta_{s\wedge\tau_\varepsilon}|^{-1}+e)^{2p}\big]\,ds + 1\\
&\le \frac{2^{p-1}}{T_0}\int_0^{t}\mathbb{E}(z_{s\wedge\tau_\varepsilon})\,ds + 2^{p-1}e^p + 1.
\end{aligned}$$

Finally

$$\mathbb{E}(z_{t\wedge\tau_\varepsilon}) \le |x-y|^{-4p} + 2^{p-1}e^p + 1 + \frac{2^{p-1}}{T_0}\int_0^t \mathbb{E}(z_{s\wedge\tau_\varepsilon})\,ds.$$

Gronwall's lemma yields that for some consatnt $C_p > 0$,

$$\mathbb{E}(z_{t\wedge\tau_\varepsilon}) \le C\,(|x-y|^{-4p}+1).$$

Letting $\varepsilon \downarrow 0$ gives the desired result. □

3 Moment estimates for two-point motions

One of the fundamental problems during 1980s was: who do we have

$$P(\tau_x = +\infty,\ \text{for all } x\in\mathbb{R}^d) = 1? \tag{3.1}$$

The condition (2.12) gives a positive response to (3.1), but as what was noticed, the coefficients still satisfy Lipschitz conditions at infinity. For the convenience of exposition, in what follows, we will denote $W_t = (W_t^1,\ldots,W_t^m)$ and

$$A_i(x) = \sigma(x)e_i \quad \text{for } i = 1,\ldots,m,$$

where $(e_1, \ldots, e_m)$ is the canonical basis of $\mathbb{R}^m$. Then the SDE (0.1) becomes

$$dX_t = \sum_{i=1}^m A_i(X_t)\, dW_t^i + b(X_t)\, dt, \quad X_0 = x. \tag{3.2}$$

When the coefficients $A_1, \ldots, A_m$ are C^2, the SDE (3.2) is equivalent to the following Stratanovich SDE:

$$dX_t = \sum_{i=1}^m A_i(X_t) \circ dW_t^i + A_0(X_t)\, dt, \quad X_0 = x, \tag{3.3}$$

with

$$A_0 = b - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^d \frac{\partial A_i}{\partial x_j} A_i^j.$$

Let $(X_t(x))$ be the solution of the Itô stochastic differential equation (3.2). The growth of the moments of $(X_t(x))$ in the spatial parameter will crucially depend on the growth behavior of the diffusion coefficients $A_1, \cdots, A_m$. In order to get the good conditions on the growth of Lipschitz constants for the estimate of $\mathbb{E}(|X_t(x) - X_t(y)|^p)$, we shall make the difference of the following conditions

$(H1)$ there are constants C_1 and $C_2 > 0$ such that

$$\sum_{i=1}^m |A_i(x)|^2 \le C_1^2, \quad |b(x)| \le C_2(1 + |x|);$$

$(H2)$ there are constants C_3 and $C_4 > 0$ such that

$$\sum_{i=1}^m |A_i(x)|^2 \le C_3^2(1 + |x|^2), \quad |b(x)| \le C_4(1 + |x|).$$

In what follows, universal positive constants appearing in the inequalities are denoted by C and allowed to change from place to place. Denote $Y_t(x) = \sup_{0 \le s \le t} |X_s(x)|$. We shall first give the explicit estimate of $||Y_1(x)||_p$ as a function of p.

Proposition 3.1 *Under the condition $(H1)$, we have for any $p > 1$,*

$$||Y_1(x)||_p \le (1 + CC_1\sqrt{p})e^{C_2}\,(1 + |x|). \tag{3.4}$$

Proof. Fix $x \in \mathbb{R}^d$. For $0 \le t \le 1$, put $\varphi(t) = ||Y_t(x)||_p$ and $M_t = \sum_{i=1}^m \int_0^t A_i(X_s(x))\, dW_s^i$. By the inequality of Burkholder, Davis and Gundy (see [10]), for any $0 \le T \le 1$,

$$\mathbb{E}\Big(\sup_{0 \le t \le T} |M_t|^p\Big) \le C\sqrt{p}^p\, \mathbb{E}\Big[\Big(\int_0^T \sum_{i=1}^m |A_i(X_s(x))|^2\, ds\Big)^{p/2}\Big] \le C\, C_1^p \sqrt{p}^p,$$

or

$$|| \sup_{0 \le t \le T} |M_t|^p||_p \le CC_1\sqrt{p}.$$

Using equation (3.2), we get the inequality

$$\varphi(T) \leq |x| + CC_1\sqrt{p} + C_2 \int_0^T (1+\varphi(s))\, ds.$$

Dividing both sides by the term $1+|x|$ and applying Gronwall's lemma to the function $\varphi(T) + 1/(1+|x|)$, we get $\frac{\varphi(1)}{1+|x|} \leq (1+CC_1\sqrt{p})\, e^{C_2}$ and the estimate (3.4) follows. □

The preceding moment inequality implies the following exponential integrability.

Corollary 3.2 *Suppose that* $(H1)$ *holds. For* $R>0$*, there is* $\delta_0 = \delta_0(C_1, C_2, R) > 0$ *such that*

$$\sup_{|x|\leq R} \mathbb{E}\Big(e^{\delta_0 Y_1^2(x)}\Big) < +\infty. \tag{3.5}$$

Proof. By (3.4), there is a constant β such that $||Y_1(x)||_p \leq \beta\sqrt{p}\,(1+|x|)$. Let $\delta > 0$. We have

$$\mathbb{E}\Big(e^{\delta Y_1^2(x)}\Big) = 1 + \sum_{p=1}^{+\infty} \frac{\delta^p \mathbb{E}(Y_1^{2p}(x))}{p!} \leq 1 + \sum_{p=1}^{+\infty} \frac{\delta^p \beta^{2p}(2p)^p(1+|x|)^{2p}}{p!}.$$

By Stirling's formula: $\dfrac{p^p}{p!} \sim \dfrac{e^p}{\sqrt{2\pi p}}$ as $p \to +\infty$. Hence, if $|x| \leq R$, the above expression is dominated by

$$C\,\Big(1 + \sum_{p=1}^{+\infty} (2\delta\beta^2\, e\,(1+R)^2)^p\Big) = \frac{C}{1-2\delta\beta^2\, e\,(1+R)^2}$$

which is finite if $\delta < 1/(2\beta^2\, e\,(1+R)^2)$. So we get (3.5). □

In the following proposition, we shall investigate estimates under (H2).

Proposition 3.3 *Under condition* $(H2)$*, there are constants* β_1 *and* $\beta_2 > 0$ *such that for all* $p \geq 2$ *and* $x \in \mathbb{R}^d$

$$||Y_1(x)||_p \leq \beta_1\, e^{\beta_2 p}\,(1+|x|). \tag{3.6}$$

Proof. Let M and φ be defined as in the proof of proposition 3.1. Under $(H2)$, we have

$$||\sup_{0\leq t\leq T} |M_t|\,||_p \leq CC_1\sqrt{p}\Big[\int_0^T (1+\varphi^2(s))\, ds\Big]^{1/2}.$$

Therefore in this case, the inequality

$$\varphi(T) \leq |x| + CC_1\sqrt{p}\Big[\int_0^T (1+\varphi^2(s))\, ds\Big]^{1/2} + C_2 \int_0^T (1+\varphi^2(s))\, ds \tag{3.7}$$

follows. To apply Gronwall's lemma, we have to square the two sides of (3.7), with the effect

$$\varphi^2(T) \leq 3\Big(|x|^2 + (C^2C_1^2 p + 2C_2^2)\int_0^T (1+\varphi^2(s))\, ds\Big).$$

It follows that

$$\frac{\varphi^2(T)+1}{(1+|x|)^2} \le 3\,\exp\{3(C^2C_1^2p+2C_2^2)T\},$$

from which we deduce (3.6). □

In the same spirit, we can treat the time variation of the one-point motion moments.

Corollary 3.4 *Under hypothesis (H1) or (H2), for any $p>1$, there is a constant $C_p>0$ (which is dependent of C_1 and C_2, or of C_3 and C_4 respectively) such that for $x\in\mathbb{R}^d, s,t\ge 0$*

$$\mathbb{E}(|X_t(x)-X_s(x)|^{2p}) \le C_p\,|t-s|^p\,(1+|x|)^{2p}. \tag{3.8}$$

Proof. We have for $s<t, x\in\mathbb{R}^d$,

$$X_t(x)-X_s(x)=\sum_{i=1}^m\int_s^t A_i(x_u(x))\,dW_u^i+\int_s^t b(x_u(x))\,du.$$

Hence there exists a constant $\beta_p>0$ such that

$$\mathbb{E}\Big(|X_t(x)-X_s(x)|^{2p}\Big) \le \beta_p\Big\{\mathbb{E}\Big[\Big(\int_s^t\sum_{i=1}^m|A_i(X_u(x))|^2\,du\Big)^p\Big]+\mathbb{E}\Big[\Big(\int_s^t|b(X_u(x))|\,du\Big)^{2p}\Big]\Big\}.$$

So we see that for some constant $C_p>0$ big enough, the right hand side of the above inequality is dominated by

$$C_p(t-s)^p(1+\mathbb{E}(Y_1(x)^{2p})).$$

Now we obtain (3.8) for an eventually different C_p by using (3.1) or (3.6). □

Now we will give precise L^p-estimates for the two-point motion under global Lipschitz conditions. Assume

(L) there exist constants L_1 and $L_2>0$ such that

$$\sum_{i=1}^N|A_i(x)-A_i(y)|^2\le L_1^2\,|x-y|^2,\quad |b(x)-b(y)|\le L_2\,|x-y|,\quad x,y\in\mathbb{R}^d.$$

Set $Y_T(x,y)=\sup_{0\le t\le T}|X_t(x)-X_t(y)|$. We shall give the explicit dependence on L_1 and L_2 for L^p- estimates of $Y_1(x,y)$.

Proposition 3.5 *Under hypothesis (L),we have for any $p>1$, all $x,y\in\mathbb{R}^d$*

$$\mathbb{E}(Y_1(x,y)^p)\le 2^p\,|x-y|^p\,e^{C\,L_1^2p^2+L_2^2p}. \tag{3.9}$$

Proof. Put $\varphi(T)=||Y_T(x,y)||_p$. As in the estimates above, we have

$$\varphi(T)\le|x-y|+C\,L_1\sqrt{p}\Big[\int_0^T\varphi^2(s)\,ds\Big]^{1/2}+L_2\int_0^T\varphi(s)\,ds. \tag{3.10}$$

Squaring the two sides of (3.10) results in

$$\varphi^2(T) \le 2\Big(2|x-y|^2 + (2C^2L_1^2 p + L_2^2)\int_0^T \varphi^2(s)\,ds\Big), \quad T \le 1.$$

It follows that for an eventually different constant $C > 0$

$$\varphi(1) \le 2\,|x-y|\,e^{CL_1^2 p + L_2^2},$$

from which we get (3.9). □

Remark. In squaring the two sides of (3.10), the control on the Lipschitz constant L_2 was lost. In order to recapture it, we shall now only consider $\mathbb{E}(|X_t(x) - X_t(y)|^p)$.

Proposition 3.6 *Assume (L). Then for any $p \ge 2$, all $x, y \in \mathbb{R}^d$, $t \in [0,1]$*

$$\mathbb{E}(|X_t(x) - X_t(y)|^{2p}) \le |x-y|^{2p}\, e^{2p^2L_1^2 + 2pL_2}. \tag{3.11}$$

Proof. Let $\xi_t = |X_t(x) - X_t(y)|^2$. By Itô's formula, we have

$$\begin{aligned} d\xi_t &= 2\sum_{i=1}^m \big\langle X_t(x) - X_t(y), A_i(X_t(x)) - A_i(X_t(y))\big\rangle\, dW_t^i \\ &\quad + 2\big\langle X_t(x) - X_t(y), b(X_t(x)) - b(X_t(y))\big\rangle\, dt \\ &\quad + \sum_{i=1}^m |A_i(X_t(x)) - A_i(X_t(y))|^2\, dt. \end{aligned}$$

The Itô stochastic contraction $d\xi_t \cdot d\xi_t$ is dominated by

$$4\sum_{i=1}^m \big\langle X_t(x) - X_t(y), A_i(X_t(x)) - A_i(X_t(y))\big\rangle^2 \le 4L_1^2\xi_t^2.$$

Again by Itô formula,

$$\begin{aligned} d\xi_t^p &= 2p\sum_{i=1}^m \xi_t^{p-1}\big\langle X_t(x) - X_t(y), A_i(X_t(x)) - A_i(X_t(y))\big\rangle\, dW_t^i \\ &\quad + 2p\xi_t^{p-1}\big\langle X_t(x) - X_t(y), b(X_t(x)) - b(x_t(y))\big\rangle\, dt \\ &\quad + p\xi_t^{p-1}\sum_{i=1}^m |A_i(X_t(x)) - A_i(X_t(y))|^2\, dt + \frac{p(p-1)}{2}\xi_t^{p-2}\, d\xi_t \cdot d\xi_t, \end{aligned}$$

which is less than

$$dM_t + (2pL_2 + 2p^2L_1^2)\,\xi_t^p\, dt$$

where M_t is the martingale part of ξ_t^p. Taking expectations, we get

$$\mathbb{E}(\xi_t^p) \le |x-y|^{2p} + (2pL_2 + 2p^2L_1^2)\int_0^t \mathbb{E}(\xi_s^p)\, ds.$$

Now Gronwall's lemma gives

$$\mathbb{E}(\xi_t^p) \le |x-y|^{2p}\, e^{2pL_2+2p^2L_1^2}, \quad t\in[0,1],$$

which is nothing but (3.11). $\square$

We shall next assume that the vector fields $b, A_1, \cdots, A_N$ are only locally Lipschitz. We shall describe growth conditions in r for the Lipschitz coefficients L_r valid on Euclidean balls of radius r that lead to L^p- moment estimates for the two-point motion of the flow. For this purpose, set

$$L_{r,1}^2 = \sum_{i=1}^m \sup_{|x|\le r} ||A_i'(x)||^2, \quad L_{r,2} = \sup_{|x|\le r} ||b'(x)|| \tag{3.12}$$

where A_i' denotes the Jacobian of the mapping $x \to A_i(x)$. Then for any $x, y \in B(r) := \{z \in \mathbb{R}^d;\ |z|\le r\}$ we have

$$\sum_{i=1}^m |A_i(x)-A_i(y)|^2 \le L_{r,1}^2\, |x-y|^2, \quad |b(x)-b(y)| \le L_{r,2}\, |x-y|.$$

Now consider a family of smooth functions $\varphi_r : \mathbb{R}^d \to \mathbb{R}$ satisfying $0\le\varphi_r\le 1$ and

$$\varphi_r(x)=1 \text{ for } |x|\le r,\ \varphi_r(x)=0 \text{ for } |x|>r+2,\ \sup_r \sup_{x\in\mathbb{R}^d} |\varphi_r'(x)|\le 1. \tag{3.13}$$

Define $A_{r,i}=\varphi_r\, A_i$ for $i=1,\cdots,m$ and $A_{r,0}=\varphi_r b$. Then we have

$$\sup_{x\in\mathbb{R}^d} |A_{r,i}'(x)|^2 \le 2\Big(\sup_{|x|\le r+2} |A_i(x)|^2 + \sup_{|x|\le r+2} ||A_i'(x)||^2\Big), \tag{3.14}$$

$$\sup_{x\in\mathbb{R}^d} |A_{r,0}'(x)| \le \sup_{|x|\le r+2} |b(x)| + \sup_{|x|\le r+2} ||b'(x)||. \tag{3.15}$$

Set

$$\tilde L_{r,1}^2 = \sum_{i=1}^m \sup_{x\in\mathbb{R}^d} ||A_{r,i}'(x)||^2, \quad \tilde L_{r,2} = \sup_{x\in\mathbb{R}^d} ||A_{r,0}'(x)||.$$

Let $(x_t^r(x))$ be the solution to the following stochastic differential equation

$$dx_t^r = \sum_{i=1}^m A_{r,i}(x_t^r)\, dW_t^i + A_{r,0}(x_t^r)\, dt, \quad x_0^r = x.$$

Applying (3.11), we get for $p\ge 2$,

$$\mathbb{E}(|x_t^r(x)-x_t^r(y)|^{2p}) \le |x-y|^{2p}\, e^{2p^2\tilde L_{r,1}^2+2p\tilde L_{r,2}}, \quad t\in[0,1]. \tag{3.16}$$

We have

$$\begin{aligned} |X_t(x)-X_t(y)|^p &= \sum_{r=1}^{+\infty} |X_t(x)-X_t(y)|^p\, \mathbf{1}_{\{r-1\le Y_1(x)\vee Y_1(y)<r\}} \\ &= \sum_{r=1}^{+\infty} |x_t^r(x)-x_t^r(y)|^p\, \mathbf{1}_{\{r-1\le Y_1(x)\vee Y_1(y)<r\}}. \end{aligned}$$

According to (3.16) and the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}(|X_t(x) - X_t(y)|^p) \leq |x-y|^p \sum_{r=1}^{+\infty} e^{p^2 \tilde{L}_{r,1}^2 + p\tilde{L}_{r,2}} \sqrt{P(Y_1(x) \vee Y_1(y) \geq r-1)}. \tag{3.17}$$

With the aid of this inequality, we are able to formulate growth conditions on the Lipschitz constants ensuring global moment estimates for the flow. In the following Theorems, this will be done consecutively under (H1) and (H2).

Theorem 3.7 *Assume (H1). Let $p \geq 2$. Suppose that $L_{r,1} \leq \alpha\, r,\ L_{r,2} \leq \beta\, r^2$. For $R > 0$, let δ_0 be given according to Corollary 3.2. Suppose*

$$p^2\alpha^2 + p\beta < \delta_0/2. \tag{3.18}$$

Then for any $R > 0$, there exists a constant $C_{p,R} > 0$ such that

$$\mathbb{E}(|X_t(x) - X_t(y)|^p) \leq C_{p,R}\, |x-y|^p, \quad \text{for } x, y \in B(R),\ t \in [0,1]. \tag{3.19}$$

In particular if for some $\varepsilon > 0$ and constants β_1, β_2 we have

$$L_{r,1} \leq \beta_1\, r^{1-\varepsilon}, \quad L_{r,2} \leq \beta_2\, r^{2-\varepsilon},$$

then for any $p \geq 2$, there exists $C_p > 0$ such that (3.19) holds.

Proof. Let $C_R = \sup_{|x| \leq R} \mathbb{E}(e^{\delta_0 Y_1^2(x)})$. Then for $r > 1$, and $x, y \in B(R)$

$$\sqrt{P(Y_1(x) \vee Y_1(y) \geq r-1)} \leq \sqrt{2C_R}\, e^{-\delta_0 (r-1)^2/2}.$$

On the other hand, by (3.14) and (3.15), we have

$$\tilde{L}_{r,1}^2 \leq 2mC_1^2 + 2\alpha^2(r+2)^2, \quad \tilde{L}_{r,2} \leq \beta r^2 + (C_2 + 2\beta)r + 3C_2 + 4\beta.$$

Therefore there exists a constant $\gamma_p > 0$, independent of r, such that

$$e^{p^2 \tilde{L}_{r,1}^2 + p\tilde{L}_{r,2}} \leq \gamma_p\, e^{(p^2\alpha^2 + p\beta)r^2}\, e^{(2\alpha^2 + C_2 + 4\beta)r}.$$

Now using (3.17), we get

$$\mathbb{E}(|X_t(x) - X_t(y)|^p) \leq \gamma_p \sqrt{2C_R}\ |x-y|^p \sum_{r=1}^{+\infty} e^{-\delta_0 (r-1)^2/2}\, e^{(p^2\alpha^2 + p\beta)r^2}\, e^{(2\alpha^2 + C_2 + 4\beta)r}.$$

It is clear that if $p^2\alpha^2 + p\beta < \delta_0/2$, the above series converges, so that (3.19) follows. □

Under (H2), the growth of the diffusion vector fields has to be counterbalanced by a slower growth of the local Lipschitz constants. We shall formulate them implicitly through conditions on the $\tilde{L}_{r,1}, \tilde{L}_{r,2}$.

Theorem 3.8 *Assume* $(H2)$ *and the existence of constants* β_1, β_2 *such that*

$$\tilde{L}_{r,1}^2 \leq \beta_1 \log r, \quad \tilde{L}_{r,2} \leq \beta_2 \log r. \tag{3.20}$$

Then for any $p \geq 2, R > 0$, *there exists a constant* $C_{p,R} > 0$ *such that*

$$\mathbb{E}(|X_t(x) - X_t(y)|^p) \leq C_{p,R}\,|x-y|^p, \quad \textit{for } x, y \in B(R),\ t \in [0,1]. \tag{3.21}$$

Proof. Let $q \geq 2$. By (3.5), $\alpha_{q,R} = \sup_{|x|\leq R} \mathbb{E}(Y_1(x)^q)$ is finite. Then for any $|x| \leq R$ and $r \geq 2$,

$$P(Y_1(x) \geq r-1) \leq \alpha_{q,R}\,\frac{1}{(r-1)^q}.$$

On the other hand, under the condition (3.20),

$$e^{p^2\tilde{L}_{r,1}^2 + p\tilde{L}_{r,2}} \leq (r+2)^{\beta_1 p^2 + p\beta_2}.$$

Therefore if we take $\frac{q}{2} > \beta_1 p^2 + \beta_2 p + 2$, the following series

$$\sum_{r\geq 2} \frac{1}{(r-1)^{q/2}} \cdot (r+2)^{\beta_1 p^2 + p\beta_2}$$

converges. Now using (3.17), we get the desired result (3.21). □

4 Limit theorems for SDE

In this section, we will use advantage the Stratonovich SDE (3.3)

$$dX_t = \sum_{i=1}^m A_i(X_t) \circ dW_t^i + A_0(X_t)\,dt, \quad X_0 = x,$$

Let $n \geq 1$ be an integer. Define $(W_t^n)_{t\in[0,1]}$ by $W_0^n = 0$ and

$$\dot{W}_t^n = 2^n\big(W_{(\ell+1)2^{-n}} - W_{\ell 2^{-n}}\big), \quad \text{for } t \in [\ell 2^{-n}, (\ell+1)2^{-n}[. \tag{4.1}$$

Let $x_t^n(x)$ be the solution to the following ordinary differential equation

$$dx_t^n = \sum_{i=1}^m A_i(x_t^n)\dot{W}_t^{n,i}\,dt + A_0(x_t^n)\,dt, \quad x_0^n = x. \tag{4.2}$$

In the classical case, where the vector fields $A_1, \cdots, A_m$ are supposed to be bounded with their first and second order bounded derivative and A_0 bounded with its bounded first derivative, almost surely, $x_t^n(x)$ converges to $X_t(x)$ uniformly with respect to (t,x) in any compact subset of $[0,1] \times \mathbb{R}^d$. The aim of this section is to establish such kind of the limit theorem without global Lipschitz conditions. We will proceed along the line of the above section.

Define $Y_n(t,x) = \sup_{0\leq s\leq t} |x_s^n(x)|$. Set

$$B_{i,k} = \sum_{j=1}^d \frac{\partial A_i}{\partial x_j} A_k^j, \quad \text{for } i = 1, \cdots, m \text{ and } k = 0, 1, \cdots, m. \tag{4.3}$$

Proposition 4.1 *Assume that*

$$\sum_{i=1}^{m} |A_i(x)|^2 \le C_1^2, \quad |A_0(x)| \le C_2(1+|x|), \tag{4.4}$$

and

$$|B_{ik}(x)| \le C_3(1+|x|) \quad \textit{for all } i,k. \tag{4.5}$$

Then there exist positive constants α_1 and α_2, independent of n and p such that

$$\mathbb{E}(Y_n(1,x)^p) \le (1+|x|)^p\, \alpha_1^p e^{\alpha_2 p^2}. \tag{4.6}$$

Proof. For $t \in [0,1]$, define $t_n = k2^{-n}$ if $t \in [k2^{-n},(k+1)2^{-n}[$ and $t_n^+ = t_n + 2^{-n}$. Then we have for fixed but arbitrary $t \in [0,1]$

$$\begin{aligned} x_t^n &= x + \sum_{i=1}^{m} \int_0^t A_i(x_{s_n}^n)\dot W_s^{n,i}\, ds + \int_0^t A_0(x_s^n)\, ds \\ &\quad + \sum_{i=1}^{m} \int_0^t \big(A_i(x_s^n) - A_i(x_{s_n}^n)\big)\dot W_s^{n,i}\, ds \\ &= x + M_n(t) + \int_0^t A_0(x_s^n)\, ds + R_n(t), \end{aligned}$$

accordingly. Consider $Y_i(s) = A_i(x_{s_n}^n)$ for $s < t_n$ and $Y_i(s) = (t-t_n)2^n A_i(x_{t_n}^n)$ for $t_n \le s \le t$. Then $M_n(t) = \sum_{i=1}^{m} \int_0^{t_n^+} Y_i(s)\, dW_s^i$. We have

$$\int_0^{t_n^+} |Y_i(s)|^2\, ds = \int_0^{t_n} |Y_i(s)|^2\, ds + 2^{-n}\,(t-t_n)^2 2^{2n} |A_i(x_{t_n}^n)|^2 \le \int_0^t |A_i(x_{s_n}^n)|^2\, ds$$

and by Burkholder's inequality

$$(i) \qquad \mathbb{E}(|M_n(t)|^p) \le C\,\sqrt{p}^p\, \mathbb{E}\Big[\Big(\int_0^{t_n^+} \sum_{i=1}^{m} |Y_i(s)|^2\, ds\Big)^{p/2}\Big] \le C C_1^p \sqrt{p}^p.$$

Remark that for n fixed, $t \to M_n(t)$ is not a martingale. Only $k \to M_n(k2^{-n})$ is a discrete martingale with respect to $\mathcal{F}_{k2^{-n}}$. Let $t \in [\ell 2^{-n}, (\ell+1)2^{-n}[$. According to (i) and by Doob's maximal inequality, we have

$$(ii) \qquad \mathbb{E}\Big(\sup_{0\le k\le \ell} |M_n(k2^{-n})|^p\Big) \le 2e\,\mathbb{E}(|M_n(t_n)|^p) \le 2eCC_1^p\sqrt{p}^p.$$

Here e is Euler's constant, resulting from the simple estimate

$$(\frac{p}{p-1})^p \le 2e, \quad p > 1.$$

Now for $s \in [k2^{-n}, (k+1)2^{-n}[$,

$$(iii) \qquad M_n(s) = M_n(k2^{-n}) + (s - k2^{-n}) \sum_{i=1}^{m} A_i(x^n_{k2^{-n}}) \big(W^i_{(k+1)2^{-n}} - W^i_{k2^{-n}}\big)\, 2^n.$$

Then $|M_n(s)| \leq |M_n(k2^{-n})| + C_1 2^{-n/2}\, \Gamma_n(k2^{-n})$, where

$$\Gamma_n(s) = 2^{n/2} \sum_{i=1}^{m} |W^i_{s_n^+} - W^i_{s_n}|. \tag{4.7}$$

Therefore

$$(iv) \qquad \sup_{0\leq s\leq t} |M_n(s)| \leq \sup_{0\leq k\leq \ell} |M_n(k2^{-n})| + C_1 \sup_{0\leq k\leq \ell} \Big(2^{-n/2}\Gamma_n(k2^{-n})\Big).$$

Now using lemma below, we have, for $p \geq 2$,

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\leq k\leq \ell} \Big(2^{-n/2}\Gamma(k2^{-n})\Big)^p\Big] &\leq \sum_k 2^{-np/2}\mathbb{E}\big(\Gamma_n(k2^{-n})^p\big) \\ &\leq 2^n \cdot 2^{-np/2}\, (Cm)^p \sqrt{p}^p \leq (Cm)^p \sqrt{p}^p. \end{aligned} \tag{4.8}$$

So combining $(iv), (ii)$ and (4.8), we finally obtain

$$|| \sup_{0\leq s\leq t} |M_n(s)|^p \,||_p \leq CC_1\sqrt{p}. \tag{4.9}$$

The remainder term R_n is more delicate to estimate. Using the vector fields defined in (4.3), we may express R_n by

$$R_n(t) = \sum_{i,k=1}^{m} \int_0^t \Big[\int_{s_n}^{s} B_{ik}(x^n_\sigma)\dot W^{n,k}_\sigma \dot W^{n,i}_s \, d\sigma\Big]\, ds + \sum_{i=1}^{m} \int_0^t \Big[\int_{s_n}^{s} B_{i0}(x^n_\sigma)\dot W^{n,i}_s \, d\sigma\Big]\, ds.$$

Let $R_{n,1}$ and $R_{n,2}$ be the two consecutive terms on the right side of the preceding equation. Using hypothesis (4.4), for $\sigma \in [s_n, s[$ we obtain

$$|x^n_\sigma| \leq |x^n_{s_n}| + C_1\, 2^{-n} \sum_{i=1}^{m} |\dot W^{n,i}_{s_n}| + C_2 \int_{s_n}^{\sigma} (1 + |x^n_s|)\, ds.$$

Hence Gronwall's lemma implies with universal constants C_1, C_2

$$1 + |x^n_\sigma| \leq \Big(|x^n_{s_n}| + 1 + C_1\, 2^{-n/2}\, \Gamma_n(s_n)\Big)\, e^{C_2 2^{-n}}. \tag{4.10}$$

Using (4.10) and hypothesis (4.5), we have

$$|R_{n,2}(t)| \leq C_3\, e^{C_2 2^{-n}} \Big[\int_0^t (|x^n_{s_n}| + 1)\Gamma_n(s_n)\, ds + C_1 \int_0^t \Gamma_n(s_n)^2\, ds\Big]. \tag{4.11}$$

By independence of $x^n_{s_n}$ and $\Gamma_n(s_n)$, we have

$$\mathbb{E}\Big((|x^n_{s_n}| + 1)^p\Gamma_n(s_n)^p\Big) \leq \mathbb{E}((1 + Y_n(s,x))^p)\, \mathbb{E}(\Gamma_n(s_n)^p). \tag{4.12}$$

Combining (4.11) and (4.12) and using (4.15) in Lemma 4.2 again, we get

$$|| \sup_{0\le s\le t} |R_{n,2}(s)| \,||_p \le C_3\, e^{C_2}\Big(Cm\sqrt{p} \int_0^t (1+||Y_n(s,x)||_p)\, ds + C_1\, C^2 m^2 p\Big). \tag{4.13}$$

In the same way

$$|| \sup_{0\le s\le t} |R_{n,1}(s)| \,||_p \le C_3\, e^{C_2}\Big(C^2 m^2 p \int_0^t (1+||Y_n(s,x)||_p)\, ds + C_1\, C^3 m^3 p^{3/2}\Big), \tag{4.14}$$

where C_3 is another universal constant, and C results from Lemma 4.2. Now denote $\psi(t) = ||Y_n(t,x)||_p$. Combining (4.9), (4.13) and (4.14), we finally obtain

$$\begin{aligned}\psi(t)+1 &\le |x|+1+CC_1\sqrt{p}+C_3\, e^{C_2}(C_1C^2m^2p+C_1C^3m^3p^{3/2})\\ &\quad + C_3\, e^{C_2}(Cm\sqrt{p}+C^2m^2p)\int_0^t (1+\psi(s))\, ds.\end{aligned}$$

From the structure of the bound just obtained we see that there are two constants $\alpha_1, \alpha_2 > 0$ independent of n and p such that $\psi(1) \le (|x|+1)\,\alpha_1\, e^{\alpha_2\, p}$ holds. The result (4.6) follows. □

Lemma 4.2 *There is a constant $C > 0$ such that*

$$||\Gamma_n(s)||_q \le Cm\sqrt{q}, \quad \textit{for all } s\in[0,1[,\ n\ge 1,\ q\ge 2. \tag{4.15}$$

Proof. Let $s \in [k2^{-n}, (k+1)2^{-n}[$ be given. Put $\gamma_i = 2^{n/2}\big(W^i_{(k+1)2^{-n}} - W^i_{k2^{-n}}\big)$. Then $\gamma_1, \cdots, \gamma_m$ are independent standard Gaussian random variables. For any $1 \le i \le m$

$$\mathbb{E}(|\gamma_i|^q) = 2\int_0^{+\infty} s^q\, e^{-s^2/2}\, \frac{ds}{\sqrt{2\pi}} = \frac{2^{q/2}}{\sqrt{\pi}} \int_0^{+\infty} s^{(q+1)/2-1}\, e^{-s}\, ds.$$

By well known properties of the Gamma function, the above quantity is dominated by $C\, q^{q/2}$ with a universal constant $C > 0$. Now

$$||\Gamma_n(s)||_q \le \sum_{i=1}^m ||\gamma_i||_{L^q} \le Cm\,\sqrt{q}.$$

We obtain (4.15). □

We next discuss the case where condition (4.4) is replaced by

$$\sum_{i=1}^m |A_i(x)|^2 \le C_1^2\,(1+|x|^2), \quad |A_0(x)| \le C_2(1+|x|). \tag{4.16}$$

(4.16) combined with (4.5) resembles (H2) of the previous section.

Proposition 4.3 *Assume (4.16) and (4.5). Then for any $p \ge 2$, there exists a constant $C_p > 0$ such that*

$$\sup_{0\le t\le 1} \mathbb{E}(|x_t^n(x)|^p) \le C_p\,(1+|x|^p), \quad \textit{for any } n\ge 1. \tag{4.17}$$

Proof. We resume the computation done in the proof of the previous Proposition, taking into account the linear growth of coefficients $A_1, \cdots, A_m$. Let $t \in [0,1]$ be fixed, and set $M_n(t) = \sum_{i=1}^m \int_0^t A_i(x_{s_n}^n) \dot{W}_s^{n,i}\, ds$. By the computations done previously, we see that for some constant $C_p > 0$

$$(i) \qquad \mathbb{E}(|M_n(t)|^p) \le C_p \int_0^t \big(1 + \mathbb{E}(|x_{s_n}^n|^p)\big)\, ds.$$

Moreover, for $\sigma \in [s_n, s_n^+[$, we have

$$|x_\sigma^n| \le |x_{s_n}^n| + C_1\Big(\int_{s_n}^\sigma (1 + |x_s^n|)\, ds\Big) \sum_{i=1}^m |\dot{W}_{s_n}^{n,i}| + C_2 \int_{s_n}^\sigma (1 + |x_s^n|)\, ds.$$

So Gronwall's lemma gives with some universal constants C_1, C_2

$$|x_\sigma^n| + 1 \le (|x_{s_n}^n| + 1)\, e^{2^{-n}(C_2 + C_1 \sum_{i=1}^m |\dot{W}_{s_n}^{n,i}|)}.$$

It follows that

$$|x_\sigma^n| + 1 \le e^{C_2}\, (|x_{s_n}^n| + 1)\, e^{C_1 \Gamma_n(s_n)}, \quad \sigma \in [s_n, s_n^+[. \tag{4.18}$$

Replacing (4.10) by (4.18) in the estimate of $R_n(t)$, we have with another universal constant C_3

$$|R_{n,2}(t)| \le C_3\, e^{C_2} \int_0^t (|x_{s_n}^n| + 1)\, e^{C_1 \Gamma_n(s_n)}\, \Gamma_n(s_n)\, ds,$$

$$|R_{n,1}(t)| \le C_3\, e^{C_2} \int_0^t (|x_{s_n}^n| + 1)\, e^{C_1 \Gamma_n(s_n)}\, \Gamma_n(s_n)^2\, ds.$$

By a direct calculation,

$$(ii) \qquad \mathbb{E}(e^{2pC_1 \Gamma_n(s)}) \le 2^m e^{4p^2 C_1^2 m/2}.$$

Now using the independence of $|x_{s_n}^n|$ and $\Gamma(s_n)$, (ii) and (4.15), we see that there is a constant $C_p > 0$ such that

$$(iii) \qquad \mathbb{E}(|R_n(t)|^p) \le C_p \int_0^t \big(1 + \mathbb{E}(|x_{s_n}^n|^p)\big)\, ds.$$

Therefore, (i) and (iii) imply

$$\mathbb{E}(|x_t^n|^p) \le C_p \Big(|x|^p + \int_0^t \big(1 + \mathbb{E}(|x_{s_n}^n|^p)\big)\, ds + \int_0^t \big(1 + \mathbb{E}(|x_s^n|^p)\big)\, ds\Big).$$

Finally consider $\psi(t) = \sup_{0 \le s \le t} \mathbb{E}(|x_s^n|^p) + 1$. The inequality just derived implies that

$$\psi(t) \le C_p\, (|x|^p + 1) + 2C_p \int_0^t \psi(s)\, ds.$$

So, a final application of Gronwall's lemma yields another constant C_p such that

$$\sup_{0 \le t \le 1} \mathbb{E}(|x_t^n|^p) \le C_p\, (1 + |x|^p).$$

The proof is completed. □

Using the same techniques, we may also derive uniform moment estimates for the time fluctuations of the approximate ordinary differential equations.

Proposition 4.4 *Assume (4.16) and (4.5). Then for any $p \geq 2$, there exists a constant $C_p > 0$, independent of n, such that*

$$\mathbb{E}(|x_s^n(x) - x_t^n(x)|^p) \leq C_p(1 + |x|^p)\,|s - t|^{p/2}. \tag{4.19}$$

Theorem 4.5 *Assume (4.16) and (4.5). Then for any $p \geq 2$ there exists a constant $C_p > 0$ such that*

$$\mathbb{E}(Y_n(1, x)^p) \leq C_p\,(1 + |x|^p), \quad \textit{for any } n \geq 1. \tag{4.20}$$

Proof. Let $\gamma > 0$ be a parameter such that $0 < \gamma < 1/2$ and $q \geq 2$ be an integer such that $2q\gamma > 1,\ 2q(\frac{1}{2} - \gamma) > 1$. Then it is known from the regularity lemma of Garsia, Rodemich and Rumsey that

$$\sup_{0 \leq t \leq 1} |\psi(t)|^{2q} \leq C_{q,\gamma} \int_0^1 \int_0^1 \frac{|\psi(s) - \psi(t)|^{2q}}{|t - s|^{1+2q\gamma}}\,dsdt.$$

Therefore we have

$$\mathbb{E}\Big(\sup_{0 \leq t \leq 1} |x_t^n(x)|^{2qp}\Big) \leq C_{q,\gamma}^p \int_0^1 \int_0^1 \frac{\mathbb{E}(|x_s^n(x) - x_t^n(x)|^{2qp})}{|t - s|^{(1+2q\gamma)p}} dsdt.$$

But by (4.19), this bound is dominated by $C_p\,(1 + |x|^p)^{2q}$, since

$$\int_0^1 \int_0^1 |t - s|^{qp-(1+2q\gamma)p}\,dsdt \leq 1.$$

So we get (4.20). □

For vector fields satisfying global Lipschitz conditions, and regularizations as considered here, Bismut [2] or Moulinier [18] proved that $\mathbb{E}(|x_t^n(x) - x_t^n(y)|^p) \leq C_p\,|x - y|^p$ for all $x, y \in \mathbb{R}^d$, where C_p is independent of n. However, the dependence of C_p on the Lipschitz continuity properties of the vector fields is not specified. In what follows, we shall make this functional dependence explicit.

Theorem 4.6 *Assume that for $x, y \in \mathbb{R}^d$*

$$\sum_{i=1}^m |A_i(x) - A_i(y)|^2 \leq L_1^2\,|x - y|^2, \quad |A_0(x) - A_0(y)| \leq L_2\,|x - y|. \tag{4.21}$$

and for all $1 \leq i \leq m, 1 \leq k \leq m$

$$|B_{ik}(x) - B_{ik}(y)| \leq K_1\,|x - y|, \quad |B_{i0}(x) - B_{i0}(y)| \leq K_2\,|x - y|. \tag{4.22}$$

Let C be the constant appearing in Lemma 4.2. Define

$$\begin{aligned}\alpha_n &= 2p((2p-1)L_1^2 + K_1)(4C^2 m^2 2^m e^{8p^2 m 2^{-n} L_1^2}) e^{2p2^{-n}L_2} \\ &\quad + 2^{-n/2} 2p((2p-1)L_1 L_2 + K_2)(2Cm2^m e^{8p^2 m 2^{-n} L_1^2}) e^{2p2^{-n}L_2}.\end{aligned}$$

Then

$$\mathbb{E}(|x_t^n(x) - x_t^n(y)|^{2p}) \leq |x - y|^{2p}\,e^{2pL_2}\,e^{\alpha_n} \leq |x - y|^{2p}\,e^{2pL_2}\,e^{\alpha_1}.$$

Proof. For n, x, y, t fixed, we have

$$x_t^n(x) - x_t^n(y) = x - y + \sum_{i=1}^m \int_0^t \big(A_i(x_s^n(x)) - A_i(x_s^n(y))\big)\dot{W}_s^{n,i}\, ds$$
$$+ \int_0^t \big(A_0(x_s^n(x)) - A_0(x_s^n(y))\big)\, ds.$$

Set $\xi_t = |x_t^n(x) - \xi_t^n(y)|^2$. Then

$$d\xi_t = 2\sum_{i=1}^m \big\langle x_t^n(x) - x_t^n(y),\ A_i(x_t^n(x)) - A_i(x_t^n(y))\big\rangle\, \dot{W}_t^{n,i}\, dt$$
$$+ 2\big\langle x_t^n(x) - x_t^n(y),\ A_0(x_t^n(x)) - A_0(x_t^n(y))\big\rangle\, dt.$$

Set

$$Q_i(t) = \big\langle x_t^n(x) - x_t^n(y),\ A_i(x_t^n(x)) - A_i(x_t^n(y))\big\rangle, \quad \text{for } i = 0, 1, \cdots, m.$$

Then $d\xi_t$ has the decomposition $d\xi_t = 2\sum_{i=1}^m Q_i(t)\dot{W}_t^{n,i}\, dt + 2Q_0(t)\, dt$. For $p \geq 2$, we have

$$\begin{aligned} d\xi_t^p &= 2p\sum_{i=1}^m \xi_t^{p-1} Q_i(t)\dot{W}_t^{n,i}\, dt + 2p\xi_t^{p-1}Q_0(t)\, dt \\ &= 2p\sum_{i=1}^m \xi_{t_n}^{p-1} Q_i(t_n)\dot{W}_t^{n,i}\, dt + 2p\xi_t^{p-1}Q_0(t)\, dt \\ &\quad + 2p\sum_{i=1}^m \Big(\xi_t^{p-1}Q_i(t) - \xi_{t_n}^{p-1}Q_i(t_n)\Big)\dot{W}_t^{n,i}\, dt. \end{aligned} \tag{4.23}$$

Let $M_t = 2p\sum_{i=1}^m \int_0^t \xi_{s_n}^{p-1} Q_i(s_n)\dot{W}_s^{n,i}\, ds$. Then $\mathbb{E}(M_t) = 0$. Moreover, we have

$$2p\int_0^t |\xi_s^{p-1}Q_0(s)|\, ds \leq 2pL_2 \int_0^t \xi_s^p\, ds. \tag{4.24}$$

To estimate the third term $R(t) = 2p\sum_{i=1}^m \int_0^t (\xi_s^{p-1}Q_i(s) - \xi_{s_n}^{p-1}Q_i(s_n))\, \dot{W}_s^{n,i}\, ds$ appearing on the right hand side of (4.23), we compute the derivative of $\xi_s^{p-1}Q_i(s)$. We get

$$\Big(\xi_s^{p-1}Q_i(s)\Big)' = (p-1)\xi_s^{p-2}\xi_s' Q_i(s) + \xi_s^{p-1}Q_i'(s).$$

Computing $Q_i'(s)$ and using our Lipschitz continuity hypotheses, we get

$$|Q_i'(s)| \leq (K_1 + L_1^2)\xi_s \sum_{k=1}^m |\dot{W}_s^{n,k}| + (L_1L_2 + K_2)\xi_s.$$

Therefore

$$\Big|\Big(\xi_s^{p-1}Q_i(s)\Big)'\Big| \leq \big((2p-1)L_1L_2 + K_2\big)\xi_s^p + \big((2p-1)L_1^2 + K_1\big)\xi_s^p \sum_{k=1}^m |\dot{W}_s^{n,k}|. \tag{4.25}$$

To estimate the contribution of ξ_s^p, note first that for $\sigma \in [s_n, s_n^+[$ we have

$$|x_\sigma^n(x) - x_\sigma^n(y)| \le |x_{s_n}^n(x) - x_{s_n}^n(y)| + L_1\Big(\int_{s_n}^{\sigma} |x_u^n(x) - x_u^n(y)|\, du\Big) \sum_{i=1}^m |\dot W_{s_n}^{n,i}| + L_2 \int_{s_n}^{\sigma} |x_u^n(x) - x_u^n(y)|\, du.$$

Now apply Gronwall's lemma. This leads to

$$|x_\sigma^n(x) - x_\sigma^n(y)| \le |x_{s_n}^n(x) - x_{s_n}^n(y)| \cdot e^{2^{-n} L_1 \sum_{i=1}^m |\dot W_{s_n}^{n,i}|}\; e^{2^{-n} L_2}.$$

Therefore for $\sigma \in [s_n, s_n^+[$,

$$\xi_\sigma^p \le \xi_{s_n}^p \cdot e^{2p 2^{-n/2} L_1 \Gamma_n(s_n)}\, e^{2p2^{-n} L_2}. \tag{4.26}$$

Hence by (4.25)

$$\begin{aligned} |R(t)| &\le 2p \sum_{i=1}^m \int_0^t \int_{s_n}^s |\big(\xi_\sigma^{p-1} Q_i(\sigma)\big)'|\, |\dot W_s^{n,i}|\, d\sigma ds \\ &\le 2p\Big\{((2p-1)L_1^2 + K_1) \int_0^t \int_{s_n}^s \xi_\sigma^p \sum_{i=1}^m |\dot W_\sigma^{n,i}| \sum_{k=1}^m |\dot W_s^{n,k}|\, d\sigma ds \\ &\quad + ((2p-1)L_1L_2 + K_2) \int_0^t \int_{s_n}^s \xi_\sigma^p \sum_{i=1}^m |\dot W_\sigma^{n,i}|\, d\sigma ds\Big\}, \end{aligned}$$

which, according to (4.26), is dominated by

$$\begin{aligned} &2p\Big\{((2p-1)L_1^2 + K_1) e^{2p2^{-n}L_2} \int_0^t \xi_{s_n}^p \Gamma_n(s_n)^2\, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\, ds \\ &\quad + 2^{-n/2}((2p-1)L_1L_2 + K_2) e^{2p2^{-n}L_2} \int_0^t \xi_{s_n}^p \Gamma_n(s_n)\, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\, ds\Big\}. \end{aligned}$$

We next employ the independence of ξ_{s_n} and $\Gamma_n(s_n)$. Therefore

$$\mathbb{E}\Big(\xi_{s_n}^p \Gamma_n(s_n)^2\, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\Big) = \mathbb{E}(\xi_{s_n}^p)\mathbb{E}\Big(\Gamma_n(s_n)^2\, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\Big).$$

By estimates derived before, using Lemma 4.2 we have

$$\mathbb{E}\Big(\Gamma_n(s_n)^2\, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\Big) \le 4C^2 m^2 2^m e^{8p^2 m 2^{-n} L_1^2},$$

and

$$\mathbb{E}\Big(\Gamma_n(s_n)\, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\Big) \le 2Cm2^m e^{8p^2 m 2^{-n} L_1^2}.$$

Summarizing, the definition

$$\begin{aligned} \alpha_n &= 2p((2p-1)L_1^2 + K_1)(4C^2 m^2 2^m e^{8p^2 m 2^{-n} L_1^2}) e^{2p2^{-n}L_2} \\ &\quad + 2^{-n/2} 2p((2p-1)L_1L_2 + K_2)(2Cm2^m e^{8p^2 m 2^{-n} L_1^2}) e^{2p2^{-n}L_2} \end{aligned} \tag{4.27}$$

implies the inequality for $\mathbb{E}(|R(t)|)$:

$$\mathbb{E}(|R(t)|) \leq \alpha_n \int_0^t \mathbb{E}(\xi_{s_n}^p)\, ds.$$

Substituting all the estimates obtained so far in (4.23), we obtain

$$\mathbb{E}(\xi_t^p) \leq |x-y|^{2p} + 2pL_2 \int_0^t \mathbb{E}(\xi_s^p)\, ds + \alpha_n \int_0^t \mathbb{E}(\xi_{s_n}^p)\, ds.$$

Finally, let $\psi_u = \sup_{0\leq s\leq u} \mathbb{E}(\xi_s^p)$. For $T > 0$ and any $0 \leq t \leq T$, the above inequality then leads to

$$\mathbb{E}(\xi_t^p) \leq |x-y|^{2p} + 2pL_2 \int_0^T \psi_s\, ds + \alpha_n \int_0^t \psi_s\, ds,$$

in other terms $\psi_T \leq |x-y|^{2p} + (2pL_2 + \alpha_n) \int_0^T \psi_s\, ds$. So Gronwall's lemma implies that for any $0 \leq t \leq 1$

$$\mathbb{E}(\xi_t^p) \leq |x-y|^{2p}\, e^{2pL_2} e^{\alpha_n}.$$

We have the desired result. □

The expression (4.27) for α_n is quite complicated. But it gives the explicit dependence of our uniform moment estimates on the Lipschitz constants for the vector fields of the underlying stochastic differential equation. We shall exploit this fact to derive a Theorem about the convergence of the ordinary differential equation regularizations given in the preceding section to the solution of the stochastic differential equation.

Let $A_1, \cdots, A_m$ be $\mathcal{C}^2$-vector fields on $\mathbb{R}^d$, A_0 is a $\mathcal{C}^1$-vector field. Suppose for $x, y \in B(n)$

$$\sum_{i=1}^m |A_i(x) - A_i(y)|^2 \leq L_{n,1}^2\, |x-y|^2, \quad |A_0(x) - A_0(y)| \leq L_{n,2}|x-y|, \tag{4.28}$$

with positive constants $L_{n,1}, L_{n,2}$. Choose a family of smooth functions $\varphi_n : \mathbb{R}^d \to \mathbb{R}$ satisfying $0 \leq \varphi_n \leq 1$ and

$$\varphi_n = 1 \text{ on } B(n),\ \varphi_n = 0 \text{ on } B(n+2)^c,\ \sup_n ||\varphi_n'||_\infty \leq 1,\ \sup_n ||\varphi_n''||_\infty \leq C < +\infty. \tag{4.29}$$

Introduce the vector fields

$$A_{n,i} = \varphi_n A_i, \quad \text{for } i = 0, 1, \cdots, m.$$

Put

$$\tilde{L}_{n,1}^2 = \sum_{i=1}^m \sup_{x\in\mathbb{R}^d} ||A_{n,i}'(x)||^2, \quad \tilde{L}_{n,2} = \sup_{x\in\mathbb{R}^d} ||A_{n,0}'(x)||. \tag{4.30}$$

Define

$$B_{ik}^n = \sum_{j=1}^d \frac{\partial A_{n,i}}{\partial x_j} A_{n,k}^j \quad \text{for } i = 1, \cdots, m \text{ and } k = 0, 1, \cdots, m,$$

and set

$$K_{n,1} = \sup_{i,k} \sup_{x\in\mathbb{R}^d} ||(B_{ik}^n)'(x)||, \quad K_{n,2} = \sup_i \sup_{x\in\mathbb{R}^d} ||(B_{i0}^n)'(x)||. \tag{4.31}$$

For $n \geq 1$, let $(z_t^n(x))$ be the solution to the following ordinary differential equation

$$dz_t^n = \sum_{i=1}^m A_{n,i}(z_t^n)\,\dot{W}_t^{n,i}\,dt + A_{n,0}(z_t^n)\,dt, \quad z_0^n = x. \tag{4.32}$$

We can apply Theorem 4.6 to obtain the estimate

$$\mathbb{E}(|z_t^n(x) - z_t^n(y)|^{2p}) \leq |x-y|^{2p}\, e^{2p\tilde{L}_{n,2}}\, e^{\tilde{\alpha}_n}, \tag{4.33}$$

where

$$\begin{aligned}\tilde{\alpha}_n &= 2p((2p-1)\tilde{L}_{n,1}^2 + K_{n,1})(4C^2m^2 2^m e^{8p^2 m 2^{-n}\tilde{L}_{n,1}^2})e^{2p2^{-n}\tilde{L}_{n,2}} \\ &\quad + 2^{-n/2}2p((2p-1)\tilde{L}_{n,1}\tilde{L}_{n,2} + K_{n,2})(2Cm2^m e^{8p^2 m 2^{-n}\tilde{L}_{n,1}^2})e^{2p2^{-n}\tilde{L}_{n,2}}.\end{aligned} \tag{4.34}$$

Now suppose that with positive constants $\tilde{\beta}_i, 1 \leq i \leq 4$, we have

$$\tilde{L}_{n,1}^2 \leq \tilde{\beta}_1 \log n,\ \ \tilde{L}_{n,2} \leq \tilde{\beta}_2 \log n,\ \ K_{n,1} \leq \tilde{\beta}_3 \log n,\ \ K_{n,2} \leq \tilde{\beta}_4 (\log n)^{3/2}. \tag{4.35}$$

Under these conditions, it is easy to see from the definition of $\tilde{\alpha}_n$ that there is a constant C_p, independent of n, such that

$$\tilde{\alpha}_n \leq C_p\,(\tilde{L}_{n,1}^2 + K_{n,1} + 1). \tag{4.36}$$

Therefore (4.33) implies

$$\mathbb{E}(|z_t^n(x) - z_t^n(y)|^{2p}) \leq |x-y|^{2p}\, e^{C_p}\, e^{2p\tilde{L}_{n,2}} e^{C_p(\tilde{L}_{n,1}^2 + K_{n,1})}.$$

Our aim is to get an estimate which is uniform relative to n. For this purpose, we shall again use the cut-off functions φ_m introduced in (4.29). For the sake of simplicity, we shall formulate conditions only on the coefficients $A_0, A_1, \cdots, A_m$. For $r \geq 1$ set

$$C_{r,1}^2 = \sum_{i=1}^m \Big(\sup_{|x|\leq r} |A_i(x)|^2\Big), \quad C_{r,2} = \sup_{|x|\leq r} |A_0(x)|,$$

$$J_{r,1} = \sup_{i,k\neq 0}\Big(\sup_{|x|\leq r} ||B'_{ik}(x)||^2\Big), \quad J_{r,2} = \sup_i \sup_{|x|\leq r} ||B'_{i0}(x)||.$$

We shall work under the following hypotheses

$$(H) \qquad \begin{cases} C_{r,1}^2 \leq \gamma_1 \log r, & C_{r,2} \leq \gamma_2 \log r, \\ L_{r,1}^2 \leq \beta_1 \log r, & L_{r,2} \leq \beta_2 \log r, \\ J_{r,1} \leq \delta_1 \log r, & J_{r,2} \leq \delta_2 (\log r)^{3/2}. \end{cases}$$

Recall that $A_{n,i} = \varphi_n\, A_i$. Under the hypothesis (H), we have

$$\sum_{i=1}^m |A_{n,i}|^2 \leq \gamma_1 \log(n+2), \quad |A_{n,0}| \leq \gamma_2 \log(n+2).$$

$$\sum_{i=1}^m ||A'_{n,i}||^2 \leq 2(\gamma_1 + \beta_1)\log(n+2), \quad ||A'_{n,0}|| \leq (\gamma_2 + \beta_2)\log(n+2).$$

Since $B^n_{ik} = \sum_{j=1}^d \frac{\partial \varphi_n}{\partial x_j} \varphi_n A_i A^j_k + \varphi_n^2 B_{ik}$, hypothesis (H) moreover implies

$$||(B^n_{ik})'|| \leq \tilde{\delta}_1 \, \log(n+2), \quad ||(B^n_{i0})'|| \leq \tilde{\delta}_2 \, (\log(n+2))^{3/2}$$

for some constants $\tilde{\delta}_1$ and $\tilde{\delta}_2$. Therefore hypothesis (H) implies conditions (4.35), so that (4.36) is validated. Now let $r \geq 1$. Consider

$$A_{r,n,i} = \varphi_r \, A_{n,i}, \quad \text{for } i = 0, 1, \cdots, m.$$

We have

$$\sum_{i=1}^m |A_{r,n,i}|^2 \leq \gamma_1 \log(r \wedge n + 2), \quad |A_{r,n,0}| \leq \gamma_2 \log(r \wedge n + 2), \tag{4.37}$$

$$\sum_{i=1}^m ||A'_{r,n,i}||^2 \leq \tilde{\beta}_1 \, \log(r \wedge n + 2), \quad ||A'_{r,n,0}|| \leq \tilde{\beta}_2 \, \log(r \wedge n + 2), \tag{4.38}$$

and

$$||(B^{rn}_{ik})'|| \leq \tilde{\delta}_1 \, \log(r \wedge n + 2), \quad ||(B^{rn}_{i0})'|| \leq \tilde{\delta}_2 \, (\log(r \wedge n + 2))^{3/2}. \tag{4.39}$$

Let $(z^{rn}_t(x))$ be the solution to

$$dz^{rn}_t = \sum_{i=1}^m A_{r,n,i}(z^{rn}_t) \, \dot{W}^{n,i}_t \, dt + A_{r,n,0}(z^{rn}_t) \, dt, \quad z^{rn}_0 = x. \tag{4.40}$$

Using (4.37)-(4.39) to estimate $\tilde{\alpha}_r$ in (4.34), we have for $r \leq n$

$$\tilde{\alpha}_r \leq C_p\big((\tilde{\beta}_1 + \tilde{\gamma}_1) \log(r+2) + 1\big).$$

We conclude

$$\begin{aligned} \mathbb{E}(|z^{rn}_t(x) - z^{rn}_t(y)|^{2p}) &\leq e^{C_p} \, e^{2p\tilde{\beta}_2 \log(r+2)} \, e^{C_p(\tilde{\beta}_1 + \tilde{\delta}_1) \log(r+2)} \, |x-y|^{2p} \\ &= e^{C_p} (r+2)^{2p\tilde{\beta}_2 + C_p(\tilde{\beta}_1 + \tilde{\delta}_1)} \, |x-y|^{2p}. \end{aligned} \tag{4.41}$$

Extrapolating in r by means of the techniques presented in section 1, we obtain the following moment estimate for the two-point motion, uniformly in the regularization parameter.

Theorem 4.7 *Under the hypothesis (H), for any $p \geq 2$ and $R > 0$, there is a constant $C_{p,R} > 0$, independent of n, such that*

$$\mathbb{E}(|z^n_t(x) - z^n_t(y)|^p) \leq C_{p,R} \, |x-y|^p, \quad \textit{for } x, y \in B(R). \tag{4.42}$$

Proof. It is clear that (H) implies the growth conditions (4.16) and (4.5). Let $Y_n(x) = \sup_{0 \leq t \leq 1} |z^n_t(x)|$. We have

$$\begin{aligned} |z^n_t(x) - z^n_t(y)|^p &= \sum_{r \geq 1} |z^n_t(x) - z^n_t(y)|^p \, \mathbf{1}_{\{r-1 \leq Y_n(x) \vee Y_n(y) < r\}} \\ &= \sum_{r \geq 1} |z^{rn}_t(x) - z^{rn}_t(y)|^p \, \mathbf{1}_{\{r-1 \leq Y_n(x) \vee Y_n(y) < r\}}. \end{aligned}$$

Let $q \geq 2$. By (4.17), there is a constant $C_{q,R} > 0$ such that for all $|x| \leq R, |y| \leq R$,

$$P(Y_n(x) \vee Y_n(y) \geq r-1) \leq C_{q,R}\frac{1}{r^q}.$$

Using (4.41), we have

$$\begin{aligned}&\mathbb{E}\Big(|z_t^{rn}(x) - z_t^{rn}(y)|^p \,\mathbf{1}_{\{r-1\leq Y_n(x)\vee Y_n(y)<r\}}\Big)\\ &\leq e^{C_p}(r+2)^{p\tilde{\beta}_2 + C_p(\tilde{\beta}_1+\tilde{\delta}_1)/2}\cdot\sqrt{C_{q,R}}\frac{1}{r^{q/2}}\,|x-y|^p.\end{aligned}$$

Now taking $q/2 \geq p\tilde{\beta}_2 + \frac{1}{2}C_p(\tilde{\beta}_1+\tilde{\delta}_1)+2$ gives (4.42). □

The following Proposition states a similar uniform moment estimate for the time fluctuations of the solutions of the regularized equations.

Proposition 4.8 *Assume hypothesis (H) is satisfied. For any $p \geq 2$ and $R > 0$, there exists a constant $C_{p,R} > 0$, independent of n, such that*

$$\mathbb{E}(|z_t^n(x) - z_s^n(x)|^p) \leq C_{p,R}\;|t-s|^{p/2}, \quad |x| \leq R,\; s,t \in [0,1]. \tag{4.43}$$

Proof. The coefficients $A_{n,i}$ and B_{ik}^n satisfy (4.5) and (4.16). So we can apply Corollary 2.4 to get (4.43). □

We are finally in a position to prove the convergence of the ordinary differential equations' regularizations (z_t^n) to the solution of the stochastic differential equation (x_t) in the L^p sense, uniformly in space and time. To state this result, we first establish it in a weaker sense.

Proposition 4.9 *Let $R > 0$ and $p \geq 2$. Then*

$$\lim_{n\to+\infty}\sup_{|x|\leq R}\sup_{0\leq t\leq 1}\mathbb{E}(|z_t^n(x) - X_t(x)|^p) = 0. \tag{4.44}$$

Proof. Let $Y_n(x) = \sup_{0\leq t\leq 1}|z_t^n(x)|$ and $Y(x) = \sup_{0\leq t\leq 1}|X_t(x)|$. Let $r \geq 1$. We have

$$\begin{aligned}\mathbb{E}(|z_t^n(x) - X_t(x)|^p) &= \mathbb{E}(|z_t^n(x) - X_t(x)|^p\,\mathbf{1}_{\{Y_n(x)\vee Y(x)\leq r\}})\\ &\quad + \mathbb{E}(|z_t^n(x) - X_t(x)|^p\,\mathbf{1}_{\{Y_n(x)\vee Y(x)>r\}}).\end{aligned}$$

Due to (3.6) and (4.20), the second term is majorized by

$$C_p\mathbb{E}((Y_n(x)^p + Y(x)^p)\,\mathbf{1}_{\{Y_n(x)\vee Y(x)>r\}}) \leq C_{p,R}\,\frac{1}{\sqrt{r}}.$$

To get (4.44), it is therefore sufficient to prove that

$$\lim_{n\to+\infty}\sup_{|x|\leq R}\sup_{0\leq t\leq 1}\mathbb{E}(|z_t^n(x) - X_t(x)|^p\mathbf{1}_{\{Y_n(x)\vee Y(x)\leq r\}}) = 0. \tag{4.45}$$

Let $n > r+2$. By uniqueness of solutions, on the subset $\{w;\ Y_n(x) \le r\}$, $z^n_t(x) = x^{nr}_t(x)$ for all $t \in [0,1]$, where $x^{nr}_t(x)$ is the solution of the following ordinary differential equation

$$dx^{nr}_t = \sum_{i=1}^m \big(\varphi_r A_i\big)(x^{nr}_t(x))\, \dot{W}^{n,i}_t\, dt + \big(\varphi_r A_0\big)(x^{nr}_t(x))\, dt, \quad x^{nr}_0 = x.$$

On the other hand, let $\tau_r(x) = \inf\{t > 0,\ |X_t(x)| \ge r\}$. Then $x_{t\wedge\tau_r(x)}(x)$ satisfies the following Itô stochastic differential equation

$$dx^r_t(x) = \sum_{i=1}^m \big(\varphi_r A_i\big)(x^r_t(x))\, dW^i_t + \Big(\varphi_r A_0 + \frac{1}{2}\sum_{i,j} \frac{\partial(\varphi_r A_i)}{\partial x_j}(\varphi_r A^j_i)\Big)\, dt, \quad x^r_0(x) = x.$$

It follows that on the subset $\{Y(x) \le r\}$ or $\{\tau_r(x) \ge 1\}$, we have $x^r_t(x) = X_t(x)$ for all $t \in [0,1]$. Therefore

$$\begin{aligned}\mathbb{E}(|z^n_t(x) - X_t(x)|^p \mathbf{1}_{\{Y_n(x)\vee Y(x)\le r\}}) &= \mathbb{E}(|x^{nr}_t(x) - x^r_t(x)|^p \mathbf{1}_{\{Y_n(x)\vee Y(x)\le r\}})\\ &\le \mathbb{E}(|x^{nr}_t(x) - x^r_t(x)|^p).\end{aligned}$$

We are now in the classical situation. Therefore Moulinier's [18] result applies to get (4.45). The proof of (4.44) is completed. □

Theorem 4.10 *Assume hypothesis (H). For any $p \ge 2$,*

$$\lim_{n\to+\infty} \mathbb{E}\Big(\sup_{0\le t\le 1}\sup_{|x|\le R} |z^n_t(x) - X_t(x)|^p\Big) = 0. \tag{4.46}$$

Proof. Let $p \ge 2$ be given. By (4.42),(4.43) and the Kolmogoroff modification theorem, there exists $\beta > 0$ such that for $|x| \le R, |y| \le R$ and $t, s \in [0,1]$,

$$|z^n_t(x) - z^n_s(y)| \le F_n \cdot \big(|x-y|^\beta + |t-s|^\beta\big), \quad n \ge 1, \tag{4.47}$$

where $\{F_n;\, n \ge 1\}$ is a family of measurable functions bounded in L^p for any p. In the same way, according to Corollary 3.4 and Theorem 3.8, there exists $F \in L^p$ such that

$$|X_t(x) - X_s(y)| \le F \cdot \big(|x-y|^\beta + |t-s|^\beta\big). \tag{4.48}$$

Let $\varepsilon_n = \sup_{0\le t\le 1}\sup_{|x|\le R} \mathbb{E}(|z^n_t(x) - X_t(x)|^p)$. By Proposition 4.9, $\lim_{n\to+\infty} \varepsilon_n = 0$. Let $\sigma_n > 0$. Then there exists $N_n \le C\,\big(\frac{1}{\sigma_n}\big)^{d+1}$ points $x_1, \cdots, x_{N_n}$ in the ball $B(R)$ and $t_1, \cdots, t_{N_n} \in [0,1]$ such that

$$[0,1] \times B(R) \subset \cup_{i=1}^{N_n} [t_i - \sigma_n, t_i + \sigma_n] \times \{x;\ |x - x_i| \le \sigma_n\}.$$

Let $(t,x) \in [0,1] \times B(R)$. There exists one i such that $|t - t_i| \le \sigma_n$ and $|x - x_i| \le \sigma_n$. We have, according to (4.47) and (4.48)

$$\begin{aligned}|z^n_t(x) - X_t(x)| &\le |z^n_t(x) - z^n_{t_i}(x_i)| + |z^n_{t_i}(x_i) - X_{t_i}(x_i)| + |X_{t_i}(x_i) - X_t(x)|\\ &\le 2(F_n + F)\sigma^\beta_n + |z^n_{t_i}(x_i) - X_{t_i}(x_i)|.\end{aligned}$$

It follows that

$$
\begin{aligned}
\sup_{0\le t\le 1}\sup_{|x|\le R}|z_t^n(x)-X_t(x)|^p &\le C_p\Big\{(F_n^p+F^p)\sigma_n^{\beta p}+\sup_{1\le i\le N_n}|z_{t_i}^n(x_i)-X_{t_i}(x_i)|^p\\
&\le C_p\Big\{(F_n^p+F^p)\sigma_n^{\beta p}+\sum_{1\le i\le N_n}|z_{t_i}^n(x_i)-X_{t_i}(x_i)|^p\Big\}
\end{aligned}
$$

with a constant C_p depending only on p. Therefore for another such constant $\hat{C}_p>0$, we have

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le 1}\sup_{|x|\le R}|z_t^n(x)-X_t(x)|^p\Big) &\le \hat{C}_p\,\sigma_n^{\beta p}+N_n\,\varepsilon_n\\
&\le \hat{C}_p\,\sigma_n^{\beta p}+C\,\big(\frac{1}{\sigma_n}\big)^{d+1}\cdot\varepsilon_n.
\end{aligned}
$$

Now taking $\sigma_n=\varepsilon_n^{1/2(d+1)}$ gives the result (4.46). □

5 Stochastic transport equations

We will begin with a determinist case. Let's consider first a vector field $V\in C_b^1(\mathbb{R}^d)$, having bounded derivative. Then the differential equation

$$
\frac{dX_t}{dt}=V(X_t),\ X_0=x \tag{5.1}
$$

defines a flow of global diffeomorphisms of $\mathbb{R}^d: x\to X_t(x)$; the inverse map $x\to X_t^{-1}(x)$ solves

$$
\frac{dX_t^{-1}}{dt}=-V(X_t^{-1}),\ X_0^{-1}=x. \tag{5.2}
$$

Now let $\theta\in C^1(\mathbb{R}^d)$. We denote by θ' the differential of θ, and $\nabla\theta$ the gradient of θ. Let $u_t=\theta(X_t^{-1})$. Then

$$
\frac{du_t}{dt}=\theta'(X_t^{-1})\cdot\frac{dX_t^{-1}}{dt}. \tag{i}
$$

On the other hand,

$$
\nabla u_t\cdot V=\theta'(X_t^{-1})\cdot D_V X_t^{-1}. \tag{ii}
$$

Now differentiating the equality $x=X_t^{-1}(X_t(x))$ with respect to the time t, we have

$$
0=\frac{dX_t^{-1}}{dt}(X_t)+(X_t^{-1})'(X_t)\frac{dX_t}{dt}=\frac{dX_t^{-1}}{dt}(X_t)+(D_V X_t^{-1})(X_t).
$$

Since X_t is bijective, so for each $x\in\mathbb{R}^d$, the above equality gives

$$
\frac{dX_t^{-1}}{dt}+D_V X_t^{-1}=0.
$$

According to (i) and (ii), we get

$$
\frac{du_t}{dt}+V\cdot\nabla u_t=0,\quad u_0=\theta\in C^1. \tag{5.3}
$$

Conversely, if $u_t \in C^1$ is a solution to (5.3), then

$$\frac{d}{dt}[u_t(X_t)] = \frac{du_t}{dt}(X_t) + u_t'(X_t)\frac{dX_t}{dt} = \frac{du_t}{dt}(X_t) + (V \cdot \nabla u_t)(X_t) = 0,$$

so that $u_t(X_t) = \theta$ or $u_t = \theta(X_t^{-1})$: it is the unique solution to (5.3).

Now let $V : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded continuous vector field such that

$$|V(x) - V(y)| \leq C|x-y| \log \frac{1}{|x-y|}, \quad \text{for } |x-y| \leq \delta < 1. \tag{5.4}$$

Then the differential equation

$$\frac{dX_t}{dt} = V(X_t), \quad X_0 = x \tag{5.5}$$

admits a unique solution $(X_t)_{t\geq 0}$, which can be constructed by Euler approximation [7]. Choose $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi \geq 0, \quad \text{supp}(\chi) \subset B(1), \quad \int_{\mathbb{R}^d} \chi dx = 1,$$

where $B(r) = \{x \in \mathbb{R}^d : |x| \leq r\}$. For $n \geq 1$, define $\chi_n(x) = 2^{dn}\chi(2^n x)$. Then $\text{supp}(\chi_n) \subset B(2^{-n})$ and $\int_{\mathbb{R}^d} \chi_n dx = 1$. Set $V_n = V * \chi_n$ (convolution product), then V_n is a bounded smooth vector field on $\mathbb{R}^d$.

Proposition 5.1 *There exists $\zeta > 1$ such that*

$$\sup_{x\in\mathbb{R}^d} |V_n(x) - V(x)| \leq \zeta^{-n} \quad \textit{for } n \textit{ big enough.} \tag{5.6}$$

Proof. We have

$$\begin{aligned}|V_n(x) - V(x)| &\leq \int_{\mathbb{R}^d} |V(x-y) - V(y)|\chi_n(y)dy \leq C\int_{B(2^{-n})} |y| \log \frac{1}{|y|} \cdot \chi_n(y)dy \\ &\leq C2^{-n} \log 2^n \cdot \int_{B(2^{-n})} \chi_n(y)dy \leq C\zeta^{-n}\end{aligned}$$

for some $\zeta > 1$. □

Theorem 5.2 *Let $X_n(t,x)_{t\geq 0}$ be the solution to*

$$\frac{dX_n}{dt} = V_n(X_n), \quad X_n(0) = x.$$

Then for any $T > 0$,

$$\lim_{n\to\infty} \sup_{t\leq T} \sup_{x\in\mathbb{R}^d} |X_n(t,x) - X(t,x)| = 0. \tag{5.7}$$

Proof. For simplicity, we omit x in X_n as well as in X. Set $\xi_n(t) = |X_n(t) - X(t)|^2$ and

$$\tau_n = \inf\{t > 0 : \xi_n(t) \geq \delta^2\}.$$

Then by (5.4) and (5.6), for $t \leq \tau_n$,

$$\left|\frac{dX_n(t)}{dt} - \frac{dX(t)}{dt}\right| \leq |V_n(X_n(t)) - V(X_n(t))| + |V(X_n(t)) - V(X(t))|$$
$$\leq \zeta^{-n} + C|X_n(t) - X(t)| \log \frac{1}{|X_n(t) - X(t)|}.$$

Therefore for $t \leq \tau_n$,

$$\left|\frac{d\xi_n(t)}{dt}\right| = 2\left|\langle X_n(t) - X(t), \frac{dX_n(t)}{dt} - \frac{dX(t)}{dt}\rangle\right| \leq 2\delta\zeta^{-n} + C\xi_n(t) \log \frac{1}{\xi_n(t)}.$$

Using the lemma below, we get for $t \leq \tau_n \wedge T$,

$$\xi_n(t) \leq (2\delta\zeta^{-n})^{e^{-Ct}} \leq (2\delta\zeta^{-n})^{e^{-CT}}.$$

This last quantity is less than δ^2 for n big enough. It follows that $\tau_n \geq T$ and

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \xi_n(t) \leq (2\delta\zeta^{-n})^{e^{-CT}}.$$

Letting $n \to \infty$ yields the result (5.7). □

A consequence of (5.7) is that $x \to X_t(x)$ defines a flow of homeomorphisms of $\mathbb{R}^d$. In fact, the inverse maps X_n^{-1} as well as X_t^{-1} satisfy the same type differential equations. In the same way, X_n^{-1} converges to X_t^{-1} uniformly with respect to (t, x) in any compact subset of $[0, +\infty[\times\mathbb{R}^d$.

Lemma 5.3 *Let $\varphi : \mathbb{R}_+ \to (0, 1)$ be a derivable function such that for $C > 0$,*

$$\varphi'(t) \leq C\varphi(t) \log \frac{1}{\varphi(t)}, \tag{5.8}$$

then

$$\varphi(t) \leq (\varphi(0))^{e^{-Ct}} \quad \textit{for } t \geq 0.$$

Proof. $\log \varphi(t)$ being negative, we use (5.8),

$$\frac{\varphi'(t)}{\varphi(t) \log \varphi(t)} \geq -C.$$

Integrating this inequality between $(0, t)$, it leads to

$$\int_{\varphi(0)}^{\varphi(t)} \frac{ds}{s \log s} \geq -Ct, \quad \text{or} \quad \log\left(\frac{\log \varphi(t)}{\log \varphi(0)}\right) \geq -Ct.$$

Therefore $\log \varphi(t) \leq \log \varphi(0) \cdot e^{-Ct}$ or $\varphi(t) \leq (\varphi(0))^{e^{-Ct}}$. □

In order to apply Lemma 5.3 in the proof of Theorem 5.2, we set

$$\eta_n(t) = \frac{2\delta}{C}\zeta^{-n} + \xi_n(t)$$

and observe that for n big enough,

$$-C\xi_n(t) \log \xi_n(t) + 2\delta\zeta^{-n} \leq -C\eta_n(t) \log \eta_n(t).$$

□

Assume now the divergence $\mathrm{div}(V) \in L^1_{loc}(\mathbb{R}^d)$ exists in the distribution sense:

$$\int_{\mathbb{R}^d} \mathrm{div}(V)\, \varphi\; dx = -\int_{\mathbb{R}^d} \nabla\varphi \cdot V\, dx, \quad \varphi \in C_c^\infty(\mathbb{R}^d). \tag{5.9}$$

We have

$$\begin{aligned}\mathrm{div}(V_n) &= \sum_{i=1}^d \frac{\partial}{\partial x_i}(V_n)^i = \sum_{i=1}^d \int_{\mathbb{R}^d} V^i(y) \frac{\partial}{\partial x_i}\chi_n(x-y)dy \\ &= -\int_{\mathbb{R}^d} V(y)\cdot \nabla_y(\chi_n(x-y))dy = \int_{\mathbb{R}^d} \mathrm{div}(V)(y)\chi_n(x-y)dy = \mathrm{div}(V) * \chi_n,\end{aligned}$$

It follows that $\mathrm{div}(V_n)$ converges to $\mathrm{div}(V)$ in $L^1_{loc}(\mathbb{R}^d)$, as $n \to +\infty$.

Theorem 5.4 *Assume* (5.4) *and* $\mathrm{div}(V)$ *exists. Let* $\theta \in C(\mathbb{R}^d)$. *Then* $u_t(x) = \theta(X_t^{-1}(x))$ *satisfies the transport equation*

$$\frac{du_t}{dt} + V\cdot\nabla u_t = 0, \quad u_0 = \theta$$

in the sense that, for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$(u_t, \varphi)_{L^2} = (\theta, \varphi)_{L^2} + \int_0^t (u_s, \mathrm{div}(\varphi V))_{L^2} ds. \tag{5.10}$$

Proof. Step 1. Suppose $\theta \in C^1$. Let X_n be given in Theorem 5.2 and set $u_n(t) = \theta(X_n^{-1}(t))$. Then for any $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$(u_n(t), \varphi)_{L^2} = (\theta, \varphi)_{L^2} + \int_0^t (u_n(s), \mathrm{div}(\varphi V_n))_{L^2} ds. \tag{5.11}$$

Let K be the support of φ, then the support of $\mathrm{div}(\varphi V_n) = \varphi\,\mathrm{div}(V_n) + \nabla\varphi\cdot V_n$ is contained in K. Let $R = \sup_{0\le t\le T}\sup_{x\in K}|X^{-1}(t,x)|$ which is finite. By (5.7), for n big enough, $X_n^{-1}(t,x) \in B(R+1)$ for all $0 \le t \le T$, $x \in K$. Therefore for $\varepsilon > 0$, there exists n_0 such that for $n \ge n_0$,

$$\sup_{x\in K}\sup_{t\le T} |\theta(X_n^{-1}(t,x)) - \theta(X^{-1}(t,x))| < \varepsilon.$$

Letting $n \to \infty$ in (5.11), we get (5.10).
Step 2. Let $\theta \in C(\mathbb{R}^d)$. Pick $\theta_n \in C^1(\mathbb{R}^d)$ such that θ_n converges to θ on any compact set. By Step 1, $u_t^n = \theta_n(X_t^{-1})$ satisfies (5.10). Now let $K = \mathrm{supp}(\varphi)$. Then $\theta_n(X_t^{-1})$ converges uniformly to $\theta(X_t^{-1})$ over K. So that letting $n \to \infty$ in

$$(u_t^n, \varphi)_{L^2} = (\theta_n, \varphi)_{L^2} + \int_0^t (u_s^n, \mathrm{div}(\varphi V))_{L^2} ds,$$

we get the result.

Corollary 5.5 *If* $div(V) = 0$, *then* X_t *preserves the Lebesgue measure.*

Proof. Take $\theta \in C_c(\mathbb{R}^d)$. Let $K = \text{supp}(\theta)$. Then $K_T = \cup_{0\le t\le T} X_t(K)$, being the image of $[0,T]\times K$ under $(t,x)\to X_t(x)$, is compact. Now for $x\in (K_T)^c$, then for any $t\in[0,T]$, $X_t^{-1}(x)\in K^c$, so that $\theta(X_t^{-1}(x)) = 0$. Now take $\varphi\in C_0^\infty(\mathbb{R}^d)$ such that $\varphi = 1$ on K_T. We have for $s\le t\le T$,

$$(u_s, \text{div}(\varphi V))_{L^2} = \int u_s(x)\,\nabla\varphi\cdot V(x)\,dx = 0,$$

so that

$$\int_{\mathbb{R}^d} \theta(X_t^{-1})dx = (u_t,\varphi)_{L^2} = (\theta,\varphi)_{L^2} = \int_{\mathbb{R}^d}\theta(x)dx,$$

which means that X_t^{-1} leaves the Lebesgue measure invariant, so does X_t. □

For the general case, we have

Theorem 5.6 *Assume* $\text{div}(V)\in L^\infty$. *Then the Lebesgue measure* λ_d *on* $\mathbb{R}^d$ *is quasi-invariant under the flow* X_t: $(X_t)_*\lambda_d = k_t\,\lambda_d$; *moreover*

$$e^{-t||\text{div}(V)||_\infty} \le k_t(x) \le e^{t||\text{div}(V)||_\infty}. \tag{5.12}$$

Proof. Take a positive function $\theta\in C_c(\mathbb{R}^d)$ and set $u_t = \theta(X_t^{-1})$. As seen in the proof of the above corollary, there exists $R>0$ such that $u_t(x) = 0$ for $t\in[0,T]$ and $|x|>R$. Then for $\varphi\in C_c^\infty(\mathbb{R}^d)$ such that $\varphi(x) = 1$ for $|x|\le R$, we have, for $s\in[0,T]$, $(u_s,\text{div}(\varphi V))_{L^2} = \int_{\mathbb{R}^d} u_s\,\text{div}(V)\,dx$. The equation (5.10) yields

$$\int_{\mathbb{R}^d} u_t\,dx = \int_{\mathbb{R}^d}\theta\,dx + \int_0^t\Big(\int_{\mathbb{R}^d} u_s\,\text{div}(V)\,dx\Big)\,ds.$$

The above equality says that $t\to\int_{\mathbb{R}^d} u_t\,dx$ is absolutely continuous and

$$\Big|\frac{d}{dt}\int_{\mathbb{R}^d} u_t\,dx\Big| \le ||\text{div}(V)||_\infty \int_{\mathbb{R}^d} u_t\,dx.$$

We deduce that

$$e^{-t||\text{div}(V)||_\infty}\int_{\mathbb{R}^d}\theta\,dx \le \int_{\mathbb{R}^d}\theta(X_t^{-1})\,dx \le e^{t||\text{div}(V)||_\infty}\int_{\mathbb{R}^d}\theta\,dx.$$

It follows that $(X_t^{-1})_*\lambda_d$ is absolutely continuous with respect to λ_d. Now (5.12) follows. □

Consider now the following Stratanovich SDE

$$dX_t = \sum_{i=1}^m A_i(X_t)\circ dW_t^i + A_0(X_t)\,dt, \quad X_0 = x. \tag{5.13}$$

When $A_1,\ldots,A_m$ are supposed to be in $C^{3+\delta}$ and $A_0\in C^{2+\delta}$, then the SDE (5.13) defines a flow of $C^{2+\delta'}$-diffeomorphisms, with $0<\delta'<\delta$. Moreover the inverse map $x\to X_t^{-1}(x)$ satisfies the SDE (see [2], p.103-106.):

$$dX_t^{-1}(x) = -\Big[(\partial_x X_t)(X_t^{-1})\Big]^{-1}\Big(\sum_{i=1}^m A_i(x)\circ dW_t^i + A_0(x)\,dt\Big). \tag{5.14}$$

For $\theta_0 \in C^{2+\delta}(\mathbb{R}^d)$, we set

$$\theta(t,x) = \theta_0(X_t^{-1}(x)). \tag{5.15}$$

Then almost surely, for each $t \geq 0$, $x \to \theta(t,x)$ is in $C^{2+\delta_0}$. We have

$$d\theta(t,x) = -\sum_{i=1}^m \big\langle \nabla\theta(t,x), A_i(x)\big\rangle \circ dW_t^i - \big\langle \nabla\theta(t,x), A_0(x)\big\rangle \tag{5.16}$$

where $d\theta(t,x)$ denotes the stochastic differential with respect to t and $\nabla\theta(t,x)$ the gradient relative to x. Passing to Itô stochastic integrals, (5.16) leads to

$$d\theta(t,x) + \sum_{i=1}^m \big\langle \nabla\theta(t,x), A_i(x)\big\rangle \cdot dW_t^i + \big\langle \nabla\theta(t,x), A_0(x)\big\rangle\, dt - \frac{1}{2}\sum_{i=1}^m (\mathcal{L}_{A_i}^2\theta)(t,x)\, dt = 0 \tag{5.17}$$

where $\mathcal{L}_{A_i}$ denotes the Lie derivative with respect to A_i.

Now suppose that A_0 satisfies the Osgood condition (5.4) and bounded. Then almost surely, for each $t > 0$, $x \to X_t(x,w)$ is a homeomorphism of $\mathbb{R}^d$.

Theorem 5.7 *Assume that* $\mathrm{div}(A_0) \in L^2_{loc}(\mathbb{R}^d)$. *Let* θ_0 *be a continuous function on* $\mathbb{R}^d$, *having polynomial growth:*

$$|\theta_0(x)| \leq C\,(1+|x|^{q_0}), \quad x \in \mathbb{R}^d. \tag{5.18}$$

Then $\theta(t,x) = \theta_0(X_t^{-1}(x))$ *solves the stochastic transport equation (5.16) in distributional sense, that is, for any* $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$(\theta_t, \phi)_{L^2} = (\theta_0,\phi)_{L^2} - \sum_{i=1}^m \int_0^t (\theta_s, \mathrm{div}(\phi A_i))_{L^2} \circ dW_s^i - \int_0^t (\theta_s, \mathrm{div}(\phi A_0))_{L^2}\, ds \tag{5.19}$$

where $\theta_t := \theta(t,\cdot)$ *and* $(\ ,\ )_{L^2}$ *denotes the inner product in* $L^2(\mathbb{R}^d, dx)$.

In order to apply (5.16), we will regularize A_0. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \chi \leq 1$, $\mathrm{supp}\chi \subset B(1)$ and $\int_{\mathbb{R}^d} \chi(x)\, dx = 1$. For $n \geq 1$, define $\chi_n(x) = 2^{dn}\chi(2^n x)$. Then $\mathrm{supp}\chi_n \subset B(2^{-n})$ and $\int_{\mathbb{R}^d} \chi_n(x)\, dx = 1$. Set

$$\bar{A}_0^n = A_0 * \chi_n \quad \text{and} \quad A_0^n = \beta_n\, \bar{A}_0^n \tag{5.20}$$

where β_n is defined such that

$$\beta_n(x) = \begin{cases} 1 \text{ if } & |x| \leq n, \\ 0 \text{ if } & |x| \geq n+1. \end{cases}$$

It is easy to see that there exists $\zeta > 1$ such that for any $M \geq 1$,

$$\sup_{|x|\leq M} |A_0^n(x) - A_0(x)| \leq \zeta^{-n} \quad \text{for } n \text{ big enough.} \tag{5.21}$$

Proposition 5.8 *Let $\{X_n(t);\ t \geq 0\}$ be the solution to the s.d.e*

$$dX_n(t) = \sum_{i=1}^{m} A_i(X_n(t)) \circ dW_t^i + A_0^n(X_n(t))\,dt, \quad X_n(0) = x. \tag{5.22}$$

Then for any $p > 1$ and $R \geq 1$,

$$\lim_{n\to+\infty} \mathbb{E}\Big(\sup_{t\leq 1}\sup_{|x|\leq R} |X_n(t) - X(t)|^p\Big) = 0. \tag{5.23}$$

Proof. We refer to [6] for a proof. □

Proof of Theorem 5.7: Fix $T \in]0,1]$ and consider $\hat{W}_t^T = W_T - W_{T-t}$ for $0 \leq t \leq T$. Let $\hat{X}_t(x, \hat{w}^T)$ be the solution to the following s.d.e

$$d\hat{X}_t = \sum_{i=1}^{m} A_i(\hat{X}_t) \circ d(\hat{W}_t^T)^i - A_0(\hat{X}_t)\,dt, \quad \hat{X}_0 = x. \tag{5.24}$$

Then $X_T^{-1}(\cdot, w) = \hat{X}_T(\cdot, \hat{w}^T)$. So some properties concerning $X_T^{-1}(\cdot, w)$ can be deduced from SDE (5.24). Replace A_0 by A_0^n in (5.24) and let $\{\hat{X}_t^n;\ 0 \leq t \leq T\}$ be the associated solution. By Proposition 5.8, for any $R \geq 1$ and $p > 1$,

$$\lim_{n\to+\infty} \mathbb{E}\Big(\sup_{t\leq 1}\sup_{|x|\leq R} |\hat{X}_t^n(x) - \hat{X}_t(x)|^p\Big) = 0.$$

Therefore for each $t \in [0,1]$,

$$\lim_{n\to+\infty} \mathbb{E}\Big(\sup_{|x|\leq R} |(X_t^n)^{-1}(x) - X_t^{-1}(x)|^p\Big) = 0. \tag{5.25}$$

Step 1. Suppose that $\theta_0 \in C_0^\infty(\mathbb{R}^d)$. Then $\theta_n(t,x) = \theta_0((X_t^n)^{-1}(x))$ solves (5.17) with A_0 replaced by A_0^n. Then for $\phi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}(\theta_n(t), \phi)_{L^2} =& (\theta_0, \phi) - \sum_{i=1}^{m} \int_0^t (\theta_n(s), \mathrm{div}(\phi A_i))_{L^2}\,dW_s^i - \int_0^t (\theta_n(s), \mathrm{div}(\phi A_0^n))_{L^2}\,ds \\ &+ \frac{1}{2}\sum_{i=1}^{m} \int_0^t \big(\theta_n(s), \mathrm{div}(A_i \mathrm{div}(\phi A_i))\big)_{L^2}\,ds.\end{aligned} \tag{5.26}$$

As $\sup_{|x|\leq R} |\theta_n(t,x) - \theta(t,x)| \leq \big(||\theta_0'||_\infty\big) \sup_{|x|\leq R} |(X_t^n)^{-1}(x) - X_t^{-1}(x)|$ and by (5.25), we have

$$\lim_{n\to+\infty} \mathbb{E}\Big(\sup_{|x|\leq R} |\theta_n(t,x) - \theta(t,x)|^p\Big) = 0. \tag{5.27}$$

Now take $R > 0$ big enough such that $\mathrm{supp}\phi \subset B(R)$. Then as $n \to +\infty$,

$$\begin{aligned}&\mathbb{E}\Big(\Big|(\theta_n(t), \phi)_{L^2} - (\theta(t), \phi)_{L^2}\Big|^2\Big) \\ &\leq \Big(\int_{\mathbb{R}^d} |\phi(x)|^2\,dx\Big)\, \mathbb{E}\Big(\int_{B(R)} |\theta_n(t,x) - \theta(t,x)|^2\,dx\Big) \to 0.\end{aligned} \tag{5.28}$$

Let $\phi_i = \mathrm{div}(\phi A_i)$ for $i = 1, \cdots, m$. Replacing ϕ by ϕ_i in (5.28), for each $s \in [0,1]$,

$$\lim_{n\to+\infty} \mathbb{E}\Big((\theta_n(s), \phi_i)_{L^2} - (\theta(s), \phi_i)_{L^2}\Big)^2 = 0.$$

On the other hand, $|(\theta_n(s), \phi_i)_{L^2}| \le ||\theta_0||_\infty \Big(\int_{\mathbb{R}^d} |\phi_i(x)|\, dx\Big)$. By Lebesgue dominated convergence theorem,

$$\lim_{n\to+\infty} \mathbb{E}\Big[\int_0^1 \sum_{i=1}^N \Big((\theta_n(s), \phi_i)_{L^2} - (\theta(s), \phi_i)_{L^2}\Big)^2\Big] = 0.$$

Therefore $\sum_{i=1}^m \int_0^t (\theta_n(s), \mathrm{div}(\phi A_i))_{L^2}\, dW_s^i$ converges to $\sum_{i=1}^m \int_0^t (\theta(s), \mathrm{div}(\phi A_i))_{L^2}\, dW_s^i$ in $L^2(\Omega)$, but uniformly relative to $t \in [0,1]$. In the same way, the last term on the right side in (5.26) tends to $\frac{1}{2}\sum_{i=1}^m \int_0^t \big(\theta(s), \mathrm{div}(A_i \mathrm{div}(\phi A_i))\big)_{L^2}\, ds$ in $L^2(\Omega)$. Now according to (5.20),

$$\mathrm{div}(\phi A_0^n) = \mathrm{div}(\phi \beta_n \bar{A}_0^n) = \beta_n\, \phi\, \mathrm{div}(A_0) * \chi_n + \big\langle \nabla(\phi\beta_n), \bar{A}_0^n \big\rangle$$

which tends to $\phi\, \mathrm{div}(A_0) + \big\langle \nabla\phi, A_0 \big\rangle = \mathrm{div}(\phi A_0)$. It follows that the term $\int_0^t (\theta_n(s), \mathrm{div}(\phi A_0^n))_{L^2}\, ds$ converges to $\int_0^t (\theta(s), \mathrm{div}(\phi A_0))_{L^2}\, ds$ in $L^2(\Omega)$. Now taking the limit $n \to +\infty$ in (5.26), we get

$$\begin{aligned}(\theta(t), \phi)_{L^2} = (\theta_0, \phi)_{L^2} &- \sum_{i=1}^m \int_0^t (\theta(s), \mathrm{div}(\phi A_i))_{L^2}\, dW_s^i - \int_0^t (\theta(s), \mathrm{div}(\phi A_0))_{L^2}\, ds \\ &+ \frac{1}{2}\sum_{i=1}^m \int_0^t \big(\theta(s), \mathrm{div}(A_i \mathrm{div}(\phi A_i))\big)_{L^2}\, ds.\end{aligned} \tag{5.29}$$

Step 2. Suppose that $\theta_0 \in C(\mathbb{R}^d)$ satisfying (5.18). Define $\theta_0^n = \beta_n\, (\theta_0 * \chi_n)$. Then there exists a constant $C > 0$ independent of n such that

$$|\theta_0^n(x)| \le C\, (1 + |x|^{q_0}), \quad x \in \mathbb{R}^d. \tag{5.30}$$

Use again the notation $\theta_n(t,x)$ to denote $\theta_n(t,x) = \theta_0^n(X_t^{-1}(x))$ where $X_t(x)$ is now the solution to the SDE (5.13). Then θ_n satisfies (5.29). Now using the SDE (5.24) and the moment estimate established in section 3, we have for any $T \in]0,1]$,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |\hat{X}_t(x)|^p\Big) \le C_p\, (1 + |x|^p),$$

in particular, for each $t \in]0,1]$,

$$\mathbb{E}\Big(|X_t^{-1}(x)|^p\Big) \le C_p\, (1 + |x|^p).$$

By (5.18) and (5.30), it holds that

$$\sup_{t\le 1} \mathbb{E}\Big(|\theta_n(t,x)|^p\Big) + \sup_{t\le 1} \mathbb{E}\Big(|\theta(t,x)|^p\Big) \le C_p\, (1 + |x|^{pq_0}). \tag{5.31}$$

Therefore for any $p > 2$, $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\mathrm{supp}\phi \subset B(R)$,

$$\int_0^1 \mathbb{E}\Big(|(\theta_n(t),\phi)_{L^2}|^p\Big)\,dt \le \Big(\int_{\mathbb{R}^d} |\phi|^q\,dx\Big)^{p-1} \int_0^1 \int_{B(R)} \mathbb{E}(|\theta_n(t,x)|^p)\,dxdt$$
$$\le C_p \Big(\int_{\mathbb{R}^d} |\phi|^q\,dx\Big)^{p-1} \int_{B(R)} (1+|x|^{pq_0})\,dx. \tag{5.32}$$

where q is the conjugate number of p. Let $M>1$. We have

$$|\theta_0^n(X_t^{-1}(x)) - \theta_0(X_t^{-1}(x))|^2 = |\theta_0^n(X_t^{-1}(x)) - \theta_0(X_t^{-1}(x))|^2\,\mathbf{1}_{(|X_t^{-1}(x)|\ge M)}$$
$$+ |\theta_0^n(X_t^{-1}(x)) - \theta_0(X_t^{-1}(x))|^2\,\mathbf{1}_{(|X_t^{-1}(x)|< M)}. \tag{5.33}$$

The second term on the right side of (5.33) is dominated by

$$\sup_{|y|\le M}\Big(|\theta_0^n(y)-\theta_0(y)|^2\Big) \to 0$$

as $n\to+\infty$. By (5.31), the expectation of the first term on the right side of (5.33) is dominated by $C_1(1+|x|^{2q_0})/M$. Therefore

$$\sup_{t\le 1}\sup_{|x|\le R} \mathbb{E}\Big(|\theta_n(t,x)-\theta(t,x)|^2\Big) \le \frac{C(1+R^{2q_0})}{M} + \sup_{|x|\le M} |\theta_0^n(x)-\theta_0(x)|^2, \quad M>1.$$

It follows that

$$\lim_{n\to+\infty}\sup_{t\le 1}\sup_{|x|\le R} \mathbb{E}\Big(|\theta_n(t,x)-\theta(t,x)|^2\Big) = 0$$

which implies that for each $\phi\in C_0^\infty(\mathbb{R}^d)$ with supp$\phi\subset B(R)$

$$\mathbb{E}\Big[\Big((\theta_n(t),\phi)_{L^2} - (\theta(t),\phi)_{L^2}\Big)^2\Big]$$
$$\le \Big(\int_{\mathbb{R}^d}|\phi|^2\,dx\Big)\int_{B(R)} \mathbb{E}(|\theta_n(t,x)-\theta(t,x)|^2)\,dx \to 0. \tag{5.34}$$

Now (5.32) and (5.34) allows to pass to the limit so that (5.19) holds true in this case. □

As an application of the above theorem, we state the following result (see [17] for a proof).

Theorem 5.9 *Assume that* div(A_0) *is bounded, then almost surely, the Lebesgue measure* λ *on* $\mathbb{R}^d$ *is quasi-invariant under the flow* X_t *and its Radon-Nikodym derivative has the following explicit expression*

$$\frac{d(X_t)_*\lambda}{d\lambda} = \exp\Big(\sum_{i=1}^m \int_0^t \mathrm{div}(A_i)(X_s(x))\circ dW_s^i + \int_0^t \mathrm{div}(A_0)(X_s(x))ds\Big). \tag{5.35}$$

6 Notes

The sections 1 and 2 are taken from [7]. For very few regular situations, we refer to the works [14] and [15].
The sections 3 and 4 are taken from [5]. A related work has been done in [16]. We refer to [12] for the discussion on strict conservativeness of SDEs.
The section 5 is taken from [6] and [17]. There are intensive works concerning Sobolev coefficients on ODE (see for example [1], [3]). Some situations on SDE were discussed in [8], [13] and [21].

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